Valuations and S-units

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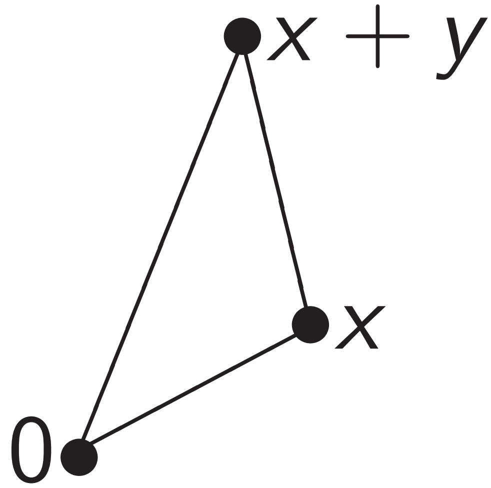
$\mathbf{R}$ = field of real numbers.
$\mathbf{C}$ = field of complex numbers.

The function $x \mapsto |x|$
from $\mathbf{C}$ to $\mathbf{R}$ is a **valuation on C**:

- $|0| = 0$.
- $x \neq 0 \Rightarrow |x| > 0$.
- $|xy| = |x||y|$.
- $|x + y| \leq |x| + |y|$.



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Not equivalent: **trivial valuation**
defined by $0 \mapsto 0$; $x \mapsto 1$ if $x \ne 0$.
Unit disk is all inputs.

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A nonequivalent nontrivial valuation on $\mathbf{Q}$: define $|0|_3 = 0$, $|x|_3 = 3^{-e_3}$ if $x = \pm 2^{e_2} 3^{e_3} 5^{e_5} \cdots$. e.g. $|90|_3 = 1/9$; $|-7/3|_3 = 3$.

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- $x \neq 0 \Rightarrow |x|_3 > 0$.
- $|xy|_3 = |x|_3 |y|_3$.
- $|x + y|_3 \leq |x|_3 + |y|_3$.

Even better: $\leq \max\{|x|_3, |y|_3\}$.

For $x \in \mathbf{Q}$, define $|x|_\infty = |x|$;
$|x|_p = p^{-e_p}$ if $x = \pm 2^{e_2} 3^{e_3} 5^{e_5} \ldots$.

x	$\lvert x\rvert_\infty$	$\lvert x\rvert_2$	$\lvert x\rvert_3$	$\lvert x\rvert_5$	$\ldots$	product
$\vdots$						
-2	2	1/2	1	1	$\ldots$	1
-1	1	1	1	1	$\ldots$	1
0	0	0	0	0	$\ldots$	0
1	1	1	1	1	$\ldots$	1
2	2	1/2	1	1	$\ldots$	1
3	3	1	1/3	1	$\ldots$	1
4	4	1/4	1	1	$\ldots$	1
5	5	1	1	1/5	$\ldots$	1
6	6	1/2	1/3	1	$\ldots$	1
$\vdots$						

[don't forget $x = 2/3$ etc.]

Infinite-dimensional lattice of $(\log|x|_\infty, \log|x|_2, \log|x|_3, \ldots)$:

$\log\|x\|_\infty$	$\log\|x\|_2$	$\log\|x\|_3$	$\log\|x\|_5$	$\ldots$
$\vdots$				
$\log 2$	$-\log 2$	0	0	$\ldots$
0	0	0	0	$\ldots$
[skip $x = 0$: log 0 not defined]				
0	0	0	0	$\ldots$
$\log 2$	$-\log 2$	0	0	$\ldots$
$\log 3$	0	$-\log 3$	0	$\ldots$
$\log 4$	$-\log 4$	0	0	$\ldots$
$\log 5$	0	0	$-\log 5$	$\ldots$
$\log 6$	$-\log 2$	$-\log 3$	0	$\ldots$
$\vdots$	[again don't forget 2/3 etc.]			

This lattice, the set of vectors $(\log|x|_\infty, \log|x|_2, \log|x|_3, \ldots)$, is
$(\log 2, -\log 2, 0, 0, 0, \ldots)\mathbf{Z} +$
$(\log 3, 0, -\log 3, 0, 0, \ldots)\mathbf{Z} +$
$(\log 5, 0, 0, -\log 5, 0, \ldots)\mathbf{Z} +$
$(\log 7, 0, 0, 0, -\log 7, \ldots)\mathbf{Z} +$
$\cdots$ where
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

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$x = \pm 2^{e_2} 3^{e_3} 5^{e_5} \cdots$ maps to
$(\log|x|_\infty, \log|x|_2, \log|x|_3, \ldots) =$
$(\log 2, -\log 2, 0, 0, 0, \ldots)e_2 +$
$(\log 3, 0, -\log 3, 0, 0, \ldots)e_3 +$
$(\log 5, 0, 0, -\log 5, 0, \ldots)e_5 +$
$(\log 7, 0, 0, 0, -\log 7, \ldots)e_7 +$
$\cdots$

Can divide $\log|x|_p$ by $\log p$ to obtain an integer "$-\operatorname{ord}_p x$"; $\operatorname{ord}_p(\pm 2^{e_2} 3^{e_3} 5^{e_5} \cdots) = e_p$.

Number theorists include the $\log p$ weight for many reasons:

- leaving out the weight would produce infinitely many short log vectors (e.g., length <2);
- want "the product formula": $\prod_v |x|_v = 1$; $\sum_v \log|x|_v = 0$;
- this particular power $|x|_v$ has a probability interpretation (matches "Haar measure" on the "completion"); etc.

Say $S \subseteq \{\infty, 2, 3, 5, \ldots\}$, $\infty \in S$.

Typical case: $p \in S \Leftrightarrow p \leq 37$.

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$\{S\text{-integers}\}$ is a subring of $\mathbf{Q}$:
closed under mult since $\mathbf{R}_{\leq 1}$ is;
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$x \in \mathbf{Q}^*$ is called an S-**unit**
if $|x|_p = 1$ for each $p \notin S$.
$\{S\text{-units}\} = \{S\text{-integers}\}^*$.

e.g. x is an $\{\infty\}$-integer

$\Leftrightarrow |x|_2 \leq 1, |x|_3 \leq 1, \ldots$

$\Leftrightarrow x \in \mathbf{Z}$.

So $\{\{\infty\}\text{-integers}\} = \mathbf{Z}$,

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$\{-1, 1\}$ have log vector $(0, 0, \ldots)$.

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$\{-1, 1\} = \mathbf{Z}^*$.
Don't confuse with $\mathbf{Q}^* = \mathbf{Q} - \{0\}$.

e.g. x is an $\{\infty, 2, 3\}$-integer
$\Leftrightarrow |x|_5 \leq 1, |x|_7 \leq 1, \ldots$
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$\Leftrightarrow$ x is "3-smooth".

For S-units can focus on S-logs:
$x \mapsto (\log |x|_\infty, \log |x|_2, \log |x|_3)$
maps group $\pm 2^{\mathbf{Z}} 3^{\mathbf{Z}}$ to lattice
$(\log 2, -\log 2, 0)\mathbf{Z} +$
$(\log 3, 0, -\log 3)\mathbf{Z}$.

Increase S for more S-units.

Prime element p of R:

- $R - pR$ closed under mult;
- $pR \neq R$ (i.e., $p \notin R^*$);
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Log: nonnegative combination of $(\log 2, -\log 2, 0, 0, \ldots)$; $(\log 3, 0, -\log 3, 0, \ldots)$; etc. u disappears in log vector.

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u logs: integer combination of
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$5^{e_5} 7^{e_7} \cdots$ logs: combine
$(\log 5, 0, 0, -\log 5, \ldots)$;
$(\log 7, 0, 0, 0, -\log 7, \ldots)$;
etc.

The 4th cyclotomic field

i: the usual $\sqrt{-1}$ in $\mathbf{C}$.

$\mathbf{Q}(i) = \mathbf{Q} + \mathbf{Q}i$ is a field:

the "field of Gaussian rationals";

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Fact: Each $x \in \mathbf{Q}(i)^*$
factors uniquely as $r \prod_{p \in P} p^{e_p}$
where $r \in \{1, i, -1, -i\}$;
$P = \{1 + i, 3, 2 + i, 2 - i, \ldots\}$;
each e_p is an integer.

$|a+bi|^2 = a^2 + b^2$ for $a, b \in \mathbf{R}$.

For each $p \in P$: have $p \in \mathbf{Z} + \mathbf{Z}i$, and $|p|^2$ is a prime not in $3 + 4\mathbf{Z}$ or the square of a prime in $3+4\mathbf{Z}$:

$p = 1 + i$: $|p|^2 = 2$.

$p = 3$: $|p|^2 = 9$.

$p = 2 + i$: $|p|^2 = 5$.

$p = 2 - i$: $|p|^2 = 5$.

$p = 7$: $|p|^2 = 49$.

$p = 11$: $|p|^2 = 121$.

$p = 3 + 2i$: $|p|^2 = 13$.

$p = 3 - 2i$: $|p|^2 = 13$.

etc. (To fully define P, also handle $1, i, -1, -i$ multiples.)

Standard *powers* of nonequivalent nontrivial valuations on $\mathbf{Q}(i)$:

$|x|_\infty = |x|^2$. (Warning: $x \mapsto |x|$ is a valuation; $x \mapsto |x|^2$ isn't!)

$|x|_{1+i} = 2^{-e_{1+i}}$.

$|x|_3 = 9^{-e_3}$. (So now $|3|_3 = 1/9$.)

$|x|_{2+i} = 5^{-e_{2+i}}$.

$|x|_{2-i} = 5^{-e_{2-i}}$.

$|x|_7 = 49^{-e_7}$.

$|x|_{11} = 121^{-e_{11}}$.

$|x|_{3+2i} = 13^{-e_{3+2i}}$.

$|x|_{3-2i} = 13^{-e_{3-2i}}$.

Etc. These have product 1.

For $x = 0$, all valuations 0.

$x \mapsto (\log|x|_\infty, \log|x|_{1+i}, \ldots)$
maps the group $\mathbf{Q}(i)^*$ onto
the infinite-dimensional lattice
$(\log 2, -\log 2, 0, 0, 0, \ldots)\mathbf{Z}$ +
$(\log 9, 0, -\log 9, 0, 0, \ldots)\mathbf{Z}$ +
$(\log 5, 0, 0, -\log 5, 0, \ldots)\mathbf{Z}$ +
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e.g. $\{\infty, 1+i, 2+i\}$-unit lattice:
$(\log 2, -\log 2, 0, 0, 0, \ldots)\mathbf{Z} +$
$(\log 5, 0, 0, -\log 5, 0, \ldots)\mathbf{Z}$.

Variant appearing in literature:
Split $|x|_\infty$ into two copies of $|x|$.
Gives slightly different lattice:

$$(0.5\log 2, 0.5\log 2, -\log 2, 0, 0, 0, \ldots)$$
$$(0.5\log 9, 0.5\log 9, 0, -\log 9, 0, 0, \ldots)$$
$$(0.5\log 5, 0.5\log 5, 0, 0, -\log 5, 0, \ldots)$$
$$(0.5\log 5, 0.5\log 5, 0, 0, 0, -\log 5, \ldots)$$
$$\vdots$$

Minor advantages: e.g., some definitions of the lattice become slightly more concise.

But now have redundant columns, each column deviating from the probability interpretation.

The 8th cyclotomic field

$\zeta_m = \exp(2\pi i/m)$ for $m \in \mathbf{Z}_{\geq 1}$.

e.g. $\zeta_8 = (1+i)/\sqrt{2}$; $\zeta_8^2 = \zeta_4 = i$.

$\mathbf{Q}(\zeta_8) = \mathbf{Q} + \mathbf{Q}\zeta_8 + \mathbf{Q}\zeta_8^2 + \mathbf{Q}\zeta_8^3$.

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Fact: Each $x \in \mathbf{Q}(\zeta_8)^*$
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where $r \in \{1, \zeta_8, \ldots, \zeta_8^7\}$;
$P = \{1+\zeta_8, 1-\zeta_8-\zeta_8^2, \ldots\}$;
$u = 1+\zeta_8+\zeta_8^2$; $e_u \in \mathbf{Z}$; $e_p \in \mathbf{Z}$.

The 8th cyclotomic field

$\zeta_m = \exp(2\pi i/m)$ for $m \in \mathbf{Z}_{\geq 1}$.
e.g. $\zeta_8 = (1+i)/\sqrt{2}$; $\zeta_8^2 = \zeta_4 = i$.
$\mathbf{Q}(\zeta_8) = \mathbf{Q} + \mathbf{Q}\zeta_8 + \mathbf{Q}\zeta_8^2 + \mathbf{Q}\zeta_8^3$.

Fact: Each $x \in \mathbf{Q}(\zeta_8)^*$
factors uniquely as $r u^{e_u} \prod_{p \in P} p^{e_p}$
where $r \in \{1, \zeta_8, \ldots, \zeta_8^7\}$;
$P = \{1+\zeta_8, 1-\zeta_8-\zeta_8^2, \ldots\}$;
$u = 1+\zeta_8+\zeta_8^2$; $e_u \in \mathbf{Z}$; $e_p \in \mathbf{Z}$.

Why isn't u included in P?
Answer: We'll want to use P to
index various nontrivial valuations.
Exercise: u valuation is trivial.

Standard valuation power ∞_1:

$|x|_{\infty_1} = |x|^2$.

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$|x|_{\infty_3} = |\sigma_3(x)|^2$ where
$\sigma_3(a_0 + a_1\zeta_8 + a_2\zeta_8^2 + a_3\zeta_8^3)$
$= a_0 + a_1\zeta_8^3 + a_2\zeta_8^6 + a_3\zeta_8^9$.
Exercise: $\sigma_3(xy) = \sigma_3(x)\sigma_3(y)$.

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To see ∞_1, ∞_3 are inequivalent:
$|1 + \zeta_8|_{\infty_1} = 2 + \sqrt{2} > 1$,
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Standard valuation for $p \in P$:
$|x|_p = N(p)^{-e_p}$, using prime
power $N(p) = |p|_{\infty_1}|p|_{\infty_3}$.

$\{\infty_1, \infty_3\}$-integers:

$\mathbf{Z}[\zeta_8] = \mathbf{Z} + \mathbf{Z}\zeta_8 + \mathbf{Z}\zeta_8^2 + \mathbf{Z}\zeta_8^3$.

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Again increase S for more S-units.

$\{\infty_1, \infty_3, 1 + \zeta_8\}$-units:
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$(1.22\ldots, -0.53\ldots, -0.69\ldots, \ldots)\mathbf{Z}$.

Reasonably short basis
for the infinite-dimensional
lattice of $\mathbf{Q}(\zeta_8)^*$ logs,
shown truncated after 2 digits:

$$\begin{matrix} 1.76 & -1.76 & 0 & 0 & 0 & \ldots \\ 1.22 & -0.53 & -0.69 & 0 & 0 & \ldots \\ 1.09 & 1.09 & 0 & -2.19 & 0 & \ldots \\ 1.09 & 1.09 & 0 & 0 & -2.19 & \ldots \\ \vdots & & & & & \end{matrix}$$

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Exercise: Find shorter basis.

The 16th cyclotomic field

$\zeta_{16} = \exp(2\pi i/16)$ so $\zeta_{16}^8 = -1$.

$$\mathbf{Q}(\zeta_{16}) = \mathbf{Q} + \mathbf{Q}\zeta_{16} + \mathbf{Q}\zeta_{16}^2 + \mathbf{Q}\zeta_{16}^3 + \mathbf{Q}\zeta_{16}^4 + \mathbf{Q}\zeta_{16}^5 + \mathbf{Q}\zeta_{16}^6 + \mathbf{Q}\zeta_{16}^7.$$

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8th roots of -1 in $\mathbf{C}$:
$\zeta_{16}^{\pm 1}, \zeta_{16}^{\pm 3}, \zeta_{16}^{\pm 5}, \zeta_{16}^{\pm 7}$.

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Each odd integer j has a unique ring morphism $\sigma_j : \mathbf{Q}(\zeta_{16}) \to \mathbf{C}$ mapping ζ_{16} to ζ_{16}^j.

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Inequivalent: $\infty_1, \infty_3, \infty_5, \infty_7$.

$\{\infty\}$-integers, meaning

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$\{\infty\}$-units: $\zeta_{16}^{\mathbf{Z}} u_1^{\mathbf{Z}} u_3^{\mathbf{Z}} u_5^{\mathbf{Z}}$ where

$u_1 = 1 + \zeta_{16} + \zeta_{16}^2$,
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Exercise: $u_1 u_3 u_5 u_7 = -1$ where

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Logs of u_1, u_3, u_5, truncated:

2.09	1.13	−2.89	−0.33
1.13	−0.33	2.09	−2.89
−2.89	2.09	−0.33	1.13

In the infinite-dimensional lattice of $\mathbf{Q}(\zeta_{16})^*$ logs, a diagonal starts after the four ∞ columns:

$$\begin{array}{rrrrrr}
2.09 & 1.13 & -2.89 & -0.33 & 0 & 0 \\
1.13 & -0.33 & 2.09 & -2.89 & 0 & 0 \\
-2.89 & 2.09 & -0.33 & 1.13 & 0 & 0 \\
1.34 & 1.01 & 0.21 & -1.88 & -0.69 & 0 \\
1.94 & -0.68 & 0.98 & 0.58 & 0 & -2.8 \\
\vdots & & & & &
\end{array}$$

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The general picture: Number of ∞ columns is between $n/2$ and n for a degree-n number field, and a diagonal appears almost immediately after the ∞ columns.