Exploring the parameter space in lattice attacks

Daniel J. Bernstein
Tanja Lange

Based on attack survey from 2019 Bernstein–Chuengsatiansup–Lange–van Vredendaal.

Some hard lattice meta-problems:

- Analyze cost of known attacks.
- Optimize attack parameters.
- Compare different attacks.
- Evaluate crypto parameters.
- Evaluate crypto designs.

`sntrup761` evaluations from "NTRU Prime: round 2" Table 2:

Ignoring cost of memory:

368	185	enum, ignoring hybrid
230	169	enum, including hybrid
153	139	sieving, ignoring hybrid
153	139	sieving, including hybrid

Accounting for cost of memory:

368	185	enum, ignoring hybrid
277	169	enum, including hybrid
208	208	sieving, ignoring hybrid
208	180	sieving, including hybrid

Security levels:

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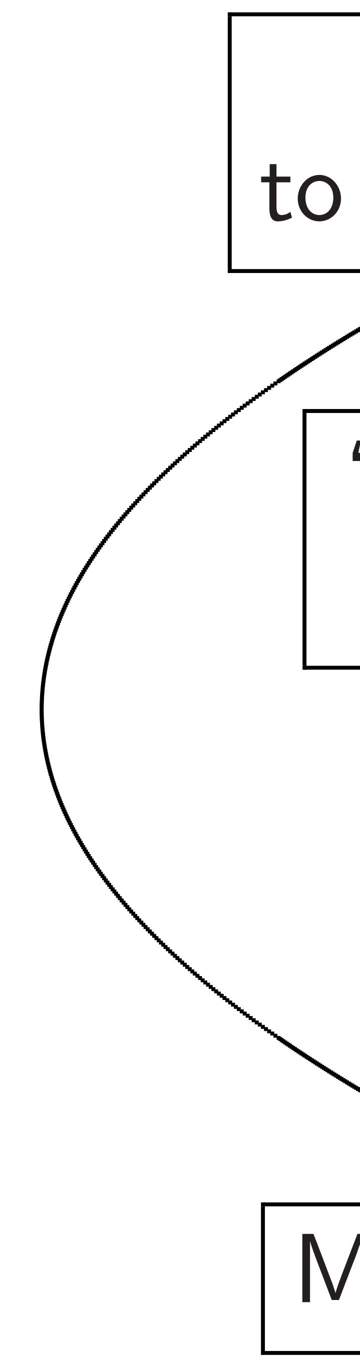

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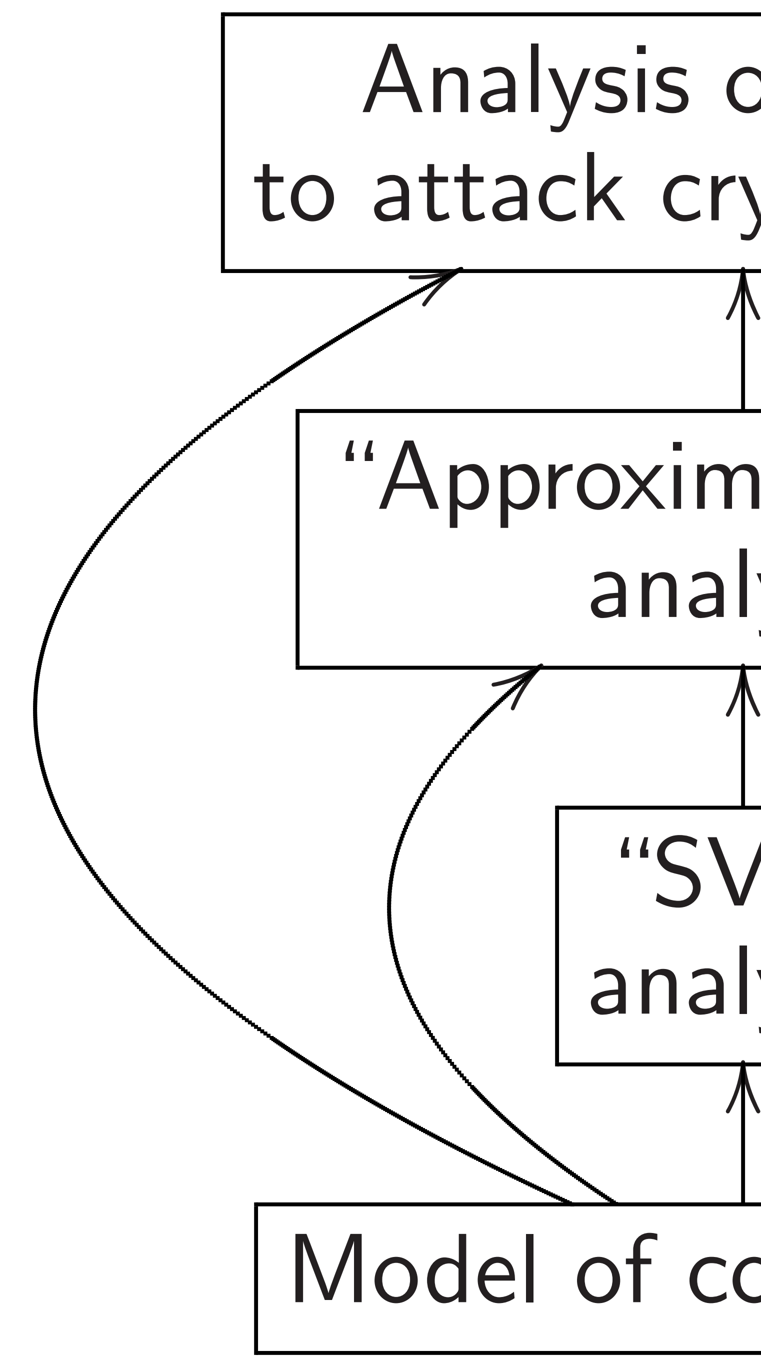

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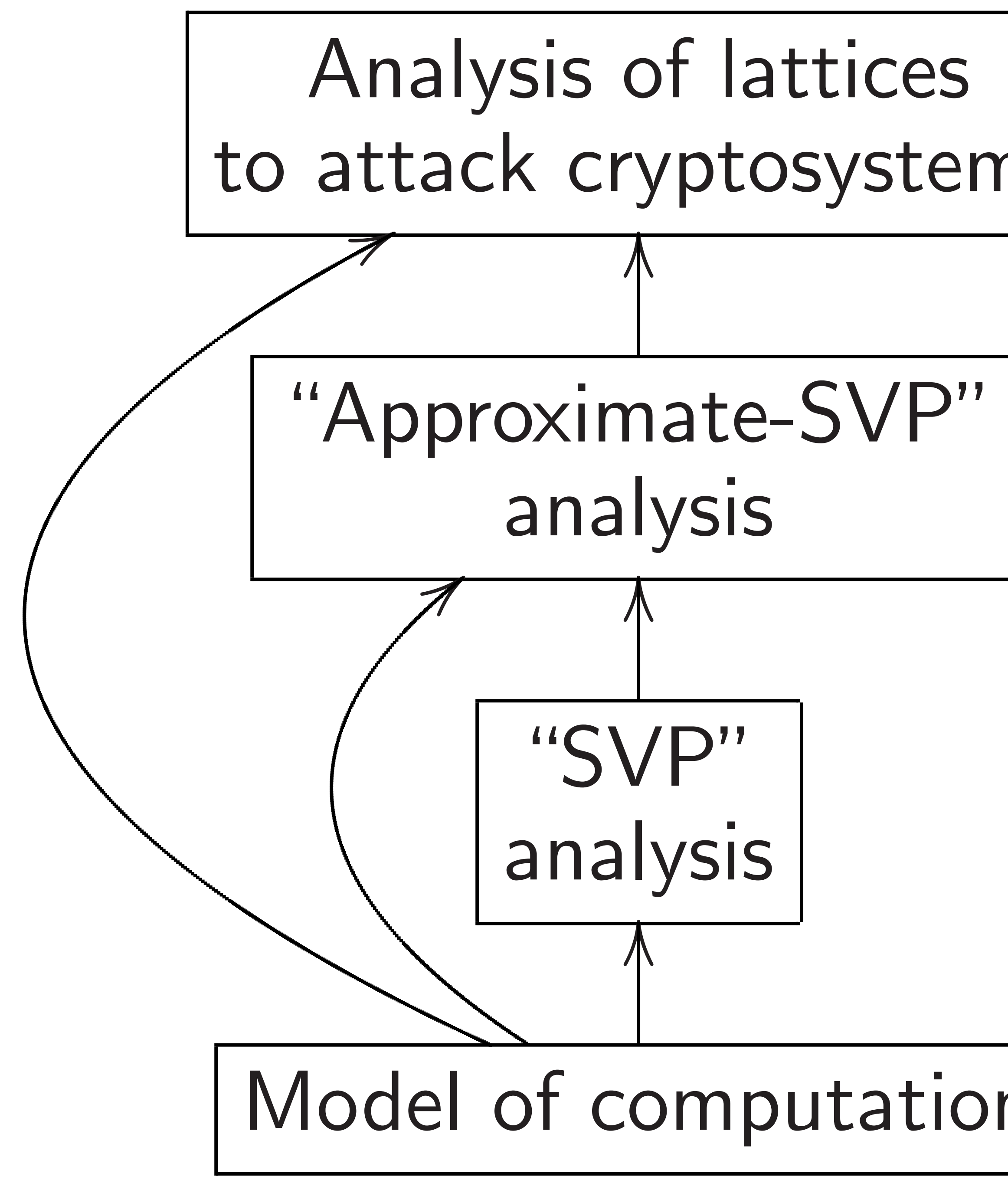

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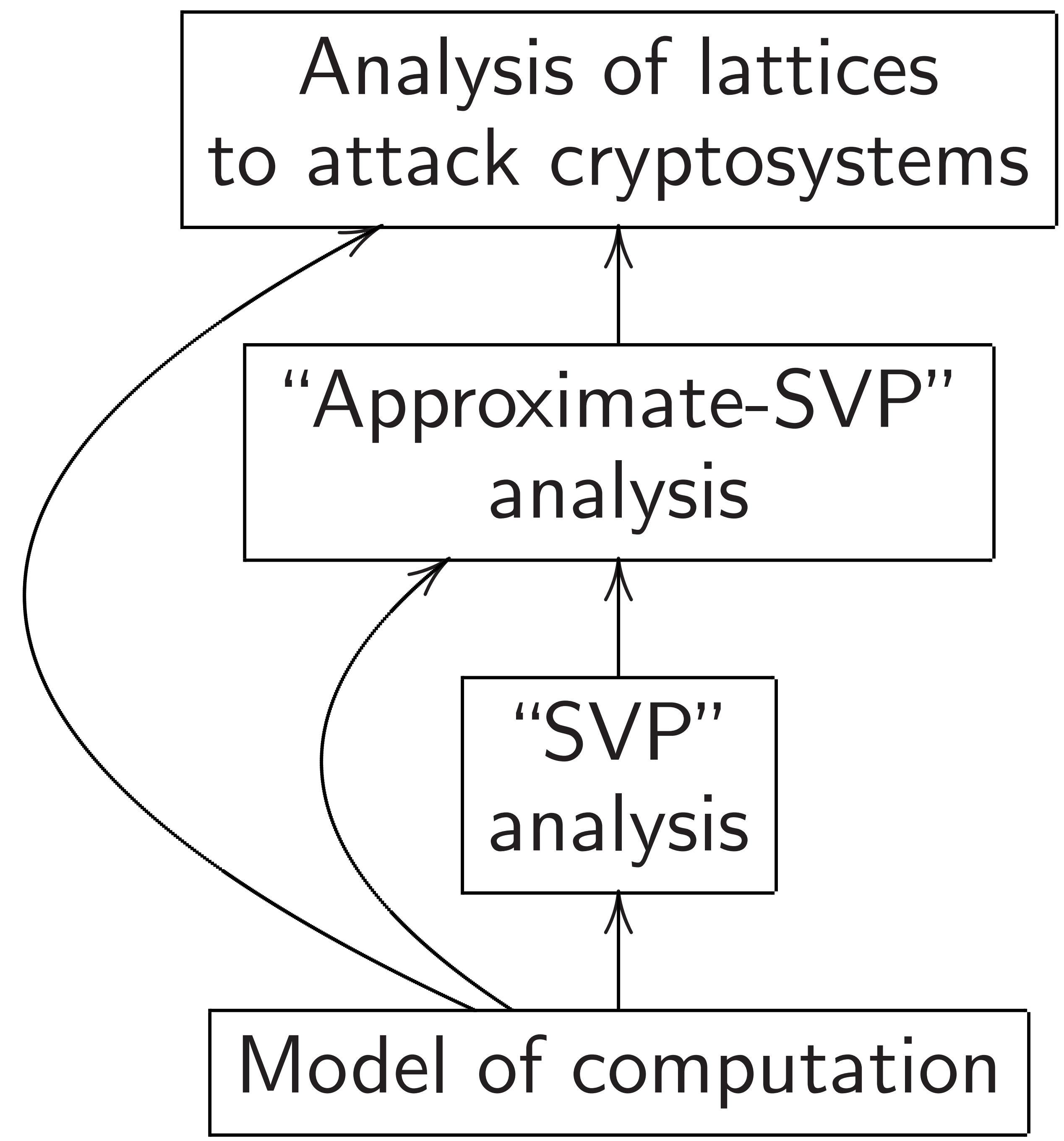

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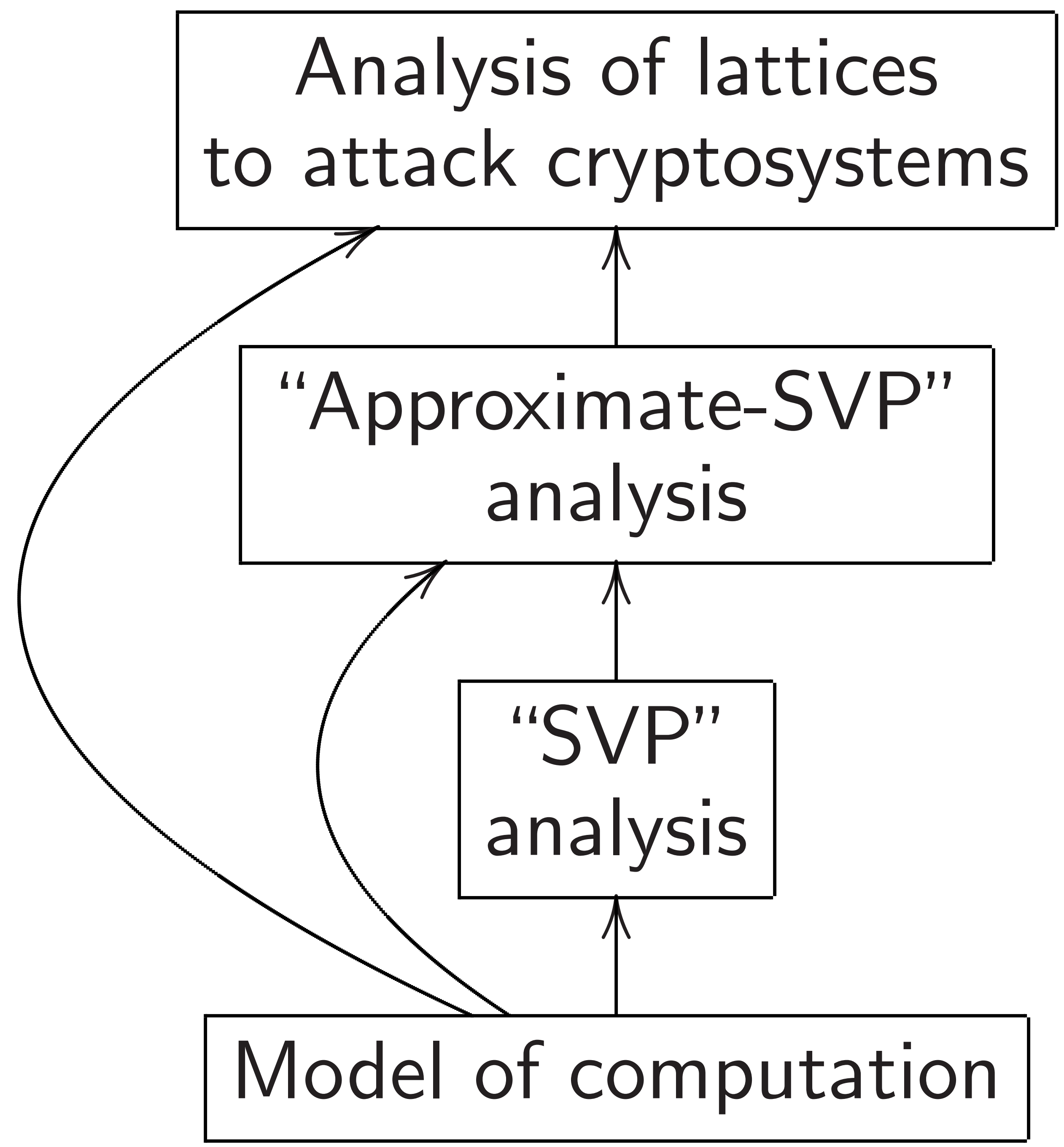

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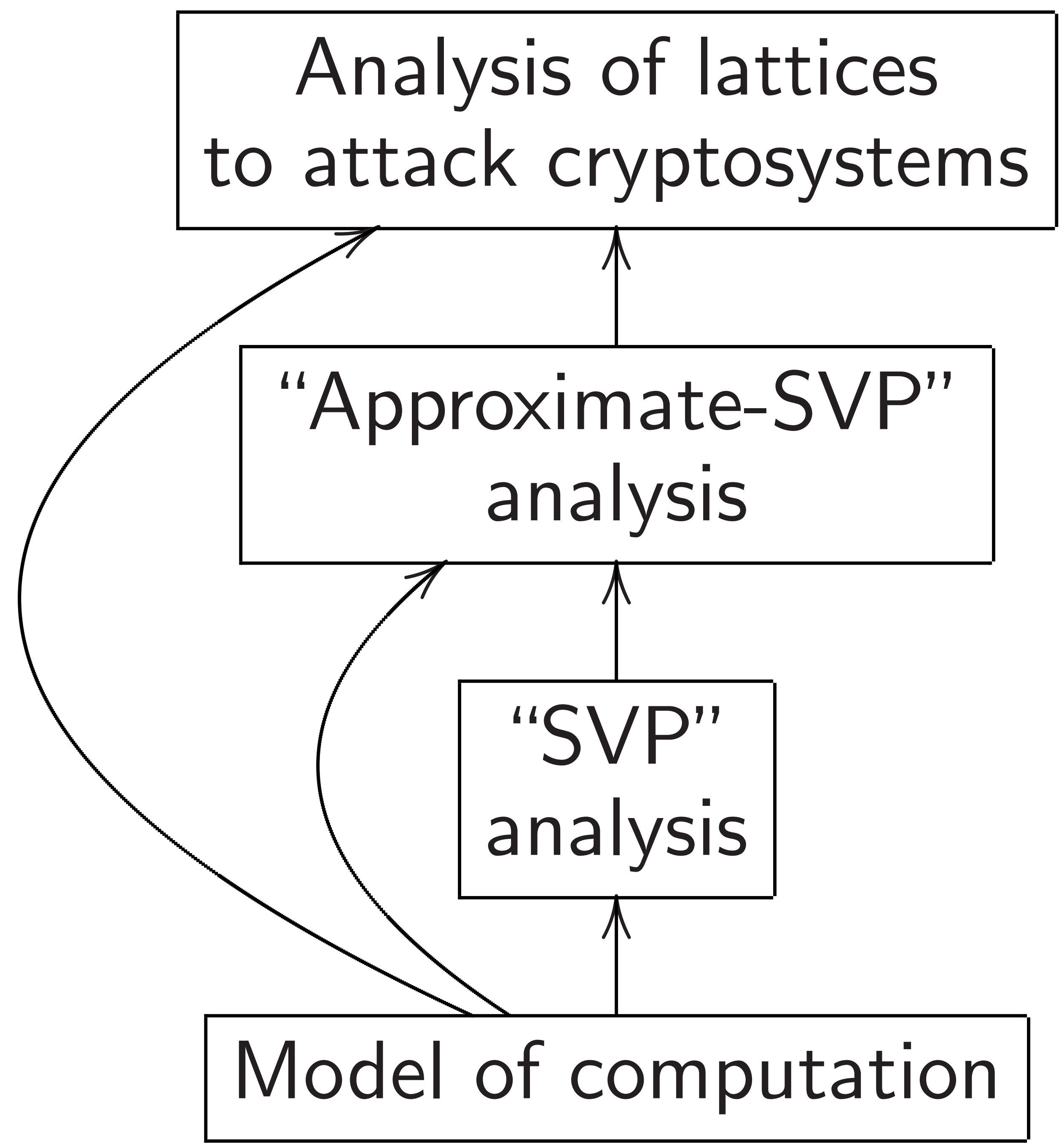

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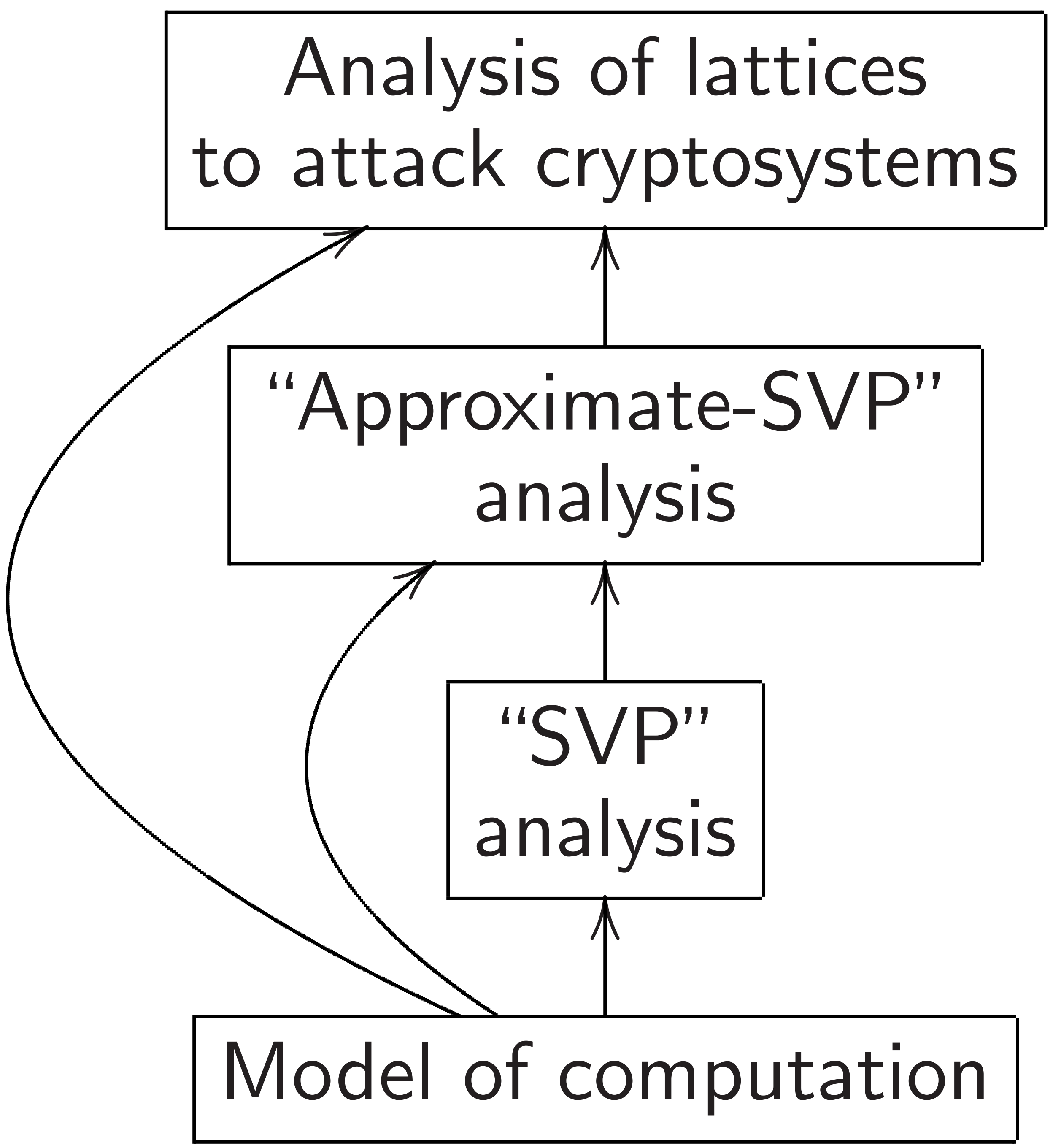

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Define $\mathcal{R} = \mathbf{Z}[x]/(x^{761} - x$
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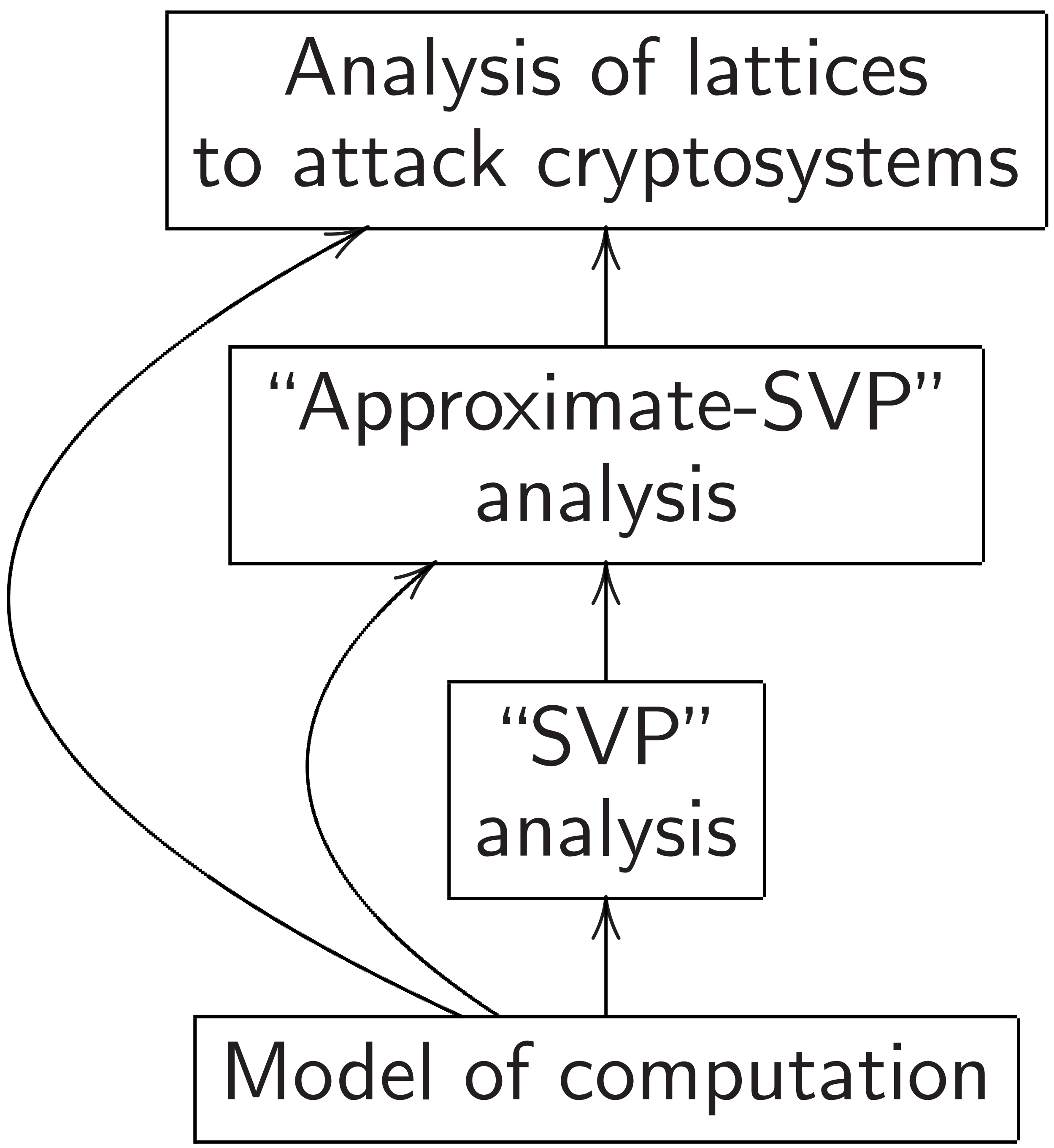

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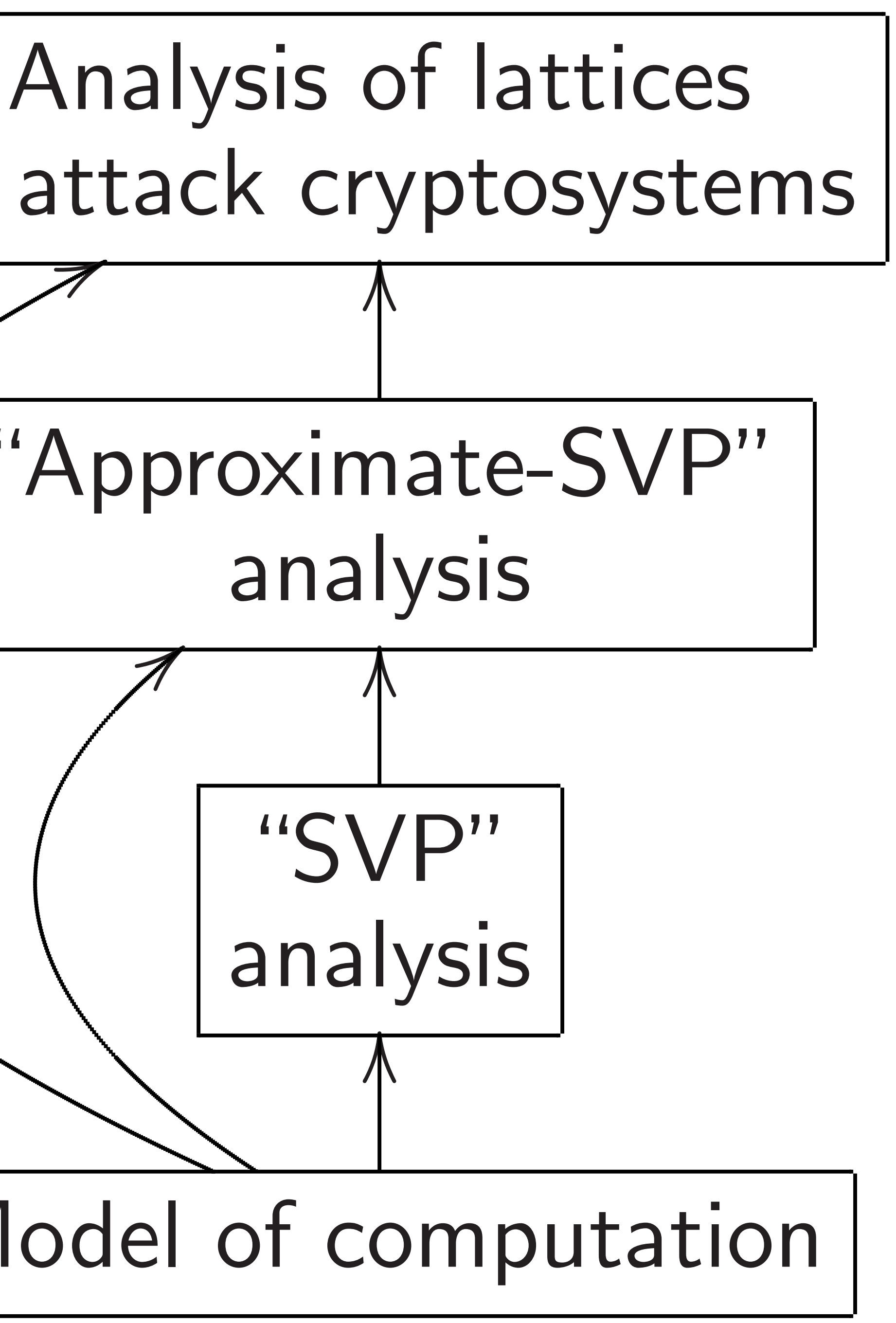

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Example

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$G = -e$

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Examples of target cryptosy

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Public key reveals multiplier
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Public key for "NTRU" (199
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$G = -e/a$, and $A = 0$.

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Recognize similarity + credits:
"NTRU" $\Rightarrow$ Quotient NTRU.
"Ring-LWE" $\Rightarrow$ Product NTRU.

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Encryption for Quotient NTRU:
Input small b, small d.
Ciphertext: $B = 3bG + d$.

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Encryption for Product NTRU:
Input encoded message M.
Randomly generate
small b, small d, small c.
Ciphertext: $B = bG + d$
and $C = bA + M + c$.

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2019 Bernstein "Comparing
proofs of security for lattice-based
encryption" includes survey of
G, a, e, c, M details and variants
in NISTPQC submissions.

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Lattices

Rewrite each problem as finding
short nonzero solution to system
of homogeneous $\mathcal{R}/q$ equations.

Problem 1: Find $(a, e) \in \mathcal{R}^2$
with $aG + e = 0$, given $G \in \mathcal{R}/q$.

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Recognize each solution spa
as a full-rank lattice:

Problem 1: Lattice is image
the map $(\overline{a}, \overline{r}) \mapsto (\overline{a}, q\overline{r} - \overline{a}$
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Lattices

Rewrite each problem as finding **short** nonzero solution to system of homogeneous $\mathcal{R}/q$ equations.

Problem 1: Find $(a, e) \in \mathcal{R}^2$ with $aG + e = 0$, given $G \in \mathcal{R}/q$.

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Problem 3: Find $(a, t_1, t_2, e_1, e_2) \in \mathcal{R}^5$ with $aG_1 + e_1 = A_1 t_1$, $aG_2 + e_2 = A_2 t_2$, given $G_1, A_1, G_2, A_2 \in \mathcal{R}/q$.

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For each of these lattices, try to figure out cost of (e.g.) BKZ-β and chance it finds short vector.

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Hybrid attacks (2008 Howgrave-Graham, . . . , 2018 Wunderer): often faster; different analysis.