# Engineering cryptographic software

Daniel J. Bernstein

University of Illinois at Chicago &
Technische Universiteit Eindhoven

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This is easy, right?

1. Take general principles of software engineering.
2. Apply principles to crypto.

Let's try some examples . . .



1972 Parnas "On the criteria to be used in decomposing systems into modules":

"We propose instead that one begins with a list of difficult design decisions or design decisions which are likely to change. Each module is then designed to hide such a decision from the others."

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#define ROUNDS 20
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/* compare the tag */
int i;
for(i = 0;i < CRYPTO_ABYTES;i++)
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## A simplified example

Target CPU: TI LM4F120H microcontroller containing one ARM Cortex-M4F core.

Reference implementation:

```
int sum(int *x)
{
  int result = 0;
  int i;
  for (i = 0;i < 1000;++i
    result += x[i];
  return result;
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Counting cycles:

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static volatile
  *const DWT_CYC
  = (void *) 0xE
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int beforesum =
int result = sum
int aftersum = *
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Counting cycles:

```
static volatile unsigned
  *const DWT_CYCCNT
  = (void *) 0xE0001004;
...

int beforesum = *DWT_CYCC
int result = sum(x);
int aftersum = *DWT_CYCCN
UARTprintf("sum %d %d\n",
  result,aftersum-befores
```

Output shows 8012 cycles.

Change 1000 to 500: 4012.

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e.g., "AI

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```
unsigned int
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0001004;

*DWT_CYCCNT;
(x);
DWT_CYCCNT;
%d %d\n",
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```

2 cycles.
00: 4012.



"Okay, 8 cycles per addition.
Um, are microcontrollers
really this slow at addition?"

Bad practice:
Apply random "optimizations"
(and tweak compiler options)
until you get bored.
Keep the fastest results.

Good practice:
Figure out lower bound for
cycles spent on arithmetic etc.
Understand gap between
lower bound and observed time.

Find "ARM Corte
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Rely on Wikipedia
M4F = M4 + float

Manual says that
"implements the A
architecture profile

Points to the "AR
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which defines instr
e.g., "ADD" for 3

First manual says
ADD takes just 1

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Find "ARM Cortex-M4 Proc
Technical Reference Manual
Rely on Wikipedia comment
M4F = M4 + floating-point

Manual says that Cortex-M4
"implements the ARMv7E-M
architecture profile".

Points to the "ARMv7-M
Architecture Reference Man
which defines instructions:
e.g., "ADD" for 32-bit addit

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Rely on Wikipedia comment that
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Inputs and output
"integer registers"
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Each element of x
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“integer registers”. ARMv7-
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special-purpose “stack point
and “program counter”.

Each element of `x` array nee
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Basic load instruction: LDR
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Inputs and output of ADD are “integer registers”. ARMv7-M has 16 integer registers, including special-purpose “stack pointer” and “program counter”.

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Each element of `x` array needs to be “loaded” into a register.

Basic load instruction: LDR. Manual says 2 cycles but adds a note about “pipelining”. Then more explanation: if next instruction is also LDR (with address not based on first LDR) then it saves 1 cycle.

$n$ consecutive LDRs takes only $n + 1$ cycles (“more multiple LDRs can b pipelined together”).

Can achieve this speed in other ways (LDRD, LDM but nothing seems faster.

Lower bound for $n$ LDR $+$ $n$ $2n + 1$ cycles, including $n$ cycles of arithme

Why observed time is higher non-consecutive LDRs; costs of manipulating `i`.

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Lower bound for $n$ LDR $+$ $n$ ADD: $2n+1$ cycles, including $n$ cycles of arithmetic.

Why observed time is higher: non-consecutive LDRs; costs of manipulating `i`.

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Lower bound for $n$ LDR + $n$ ADD:
$2n+1$ cycles,
including $n$ cycles of arithmetic.

Why observed time is higher:
non-consecutive LDRs;
costs of manipulating `i`.

```
int sum
{
  int r
  int *
  int x
      x
  while
    x0
    x1
    x2
    x3
    x4
    x5
    x6
```

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including $n$ cycles of arithmetic.

Why observed time is higher:
non-consecutive LDRs;
costs of manipulating `i`.

```
int sum(int *x)
{
  int result = 0
  int *y = x + 1
  int x0,x1,x2,x
      x5,x6,x7,x

  while (x != y)
    x0 = 0[(vola
    x1 = 1[(vola
    x2 = 2[(vola
    x3 = 3[(vola
    x4 = 4[(vola
    x5 = 5[(vola
    x6 = 6[(vola
```

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takes only $n+1$ cycles
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$2n+1$ cycles,
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Why observed time is higher:
non-consecutive LDRs;
costs of manipulating `i`.

```
int sum(int *x)
{
  int result = 0;
  int *y = x + 1000;
  int x0,x1,x2,x3,x4,
      x5,x6,x7,x8,x9;

  while (x != y) {
    x0 = 0[(volatile int
    x1 = 1[(volatile int
    x2 = 2[(volatile int
    x3 = 3[(volatile int
    x4 = 4[(volatile int
    x5 = 5[(volatile int
    x6 = 6[(volatile int
```

$n$ consecutive LDRs
takes only $n+1$ cycles
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Lower bound for $n$ LDR + $n$ ADD:
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int sum(int *x)
{
  int result = 0;
  int *y = x + 1000;
  int x0,x1,x2,x3,x4,
      x5,x6,x7,x8,x9;

  while (x != y) {
    x0 = 0[(volatile int *)x];
    x1 = 1[(volatile int *)x];
    x2 = 2[(volatile int *)x];
    x3 = 3[(volatile int *)x];
    x4 = 4[(volatile int *)x];
    x5 = 5[(volatile int *)x];
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```

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multiple LDRs can be
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served time is higher:
secutive LDRs;
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```
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  int result = 0;
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    x3 = 3[(volatile int *)x];
    x4 = 4[(volatile int *)x];
    x5 = 5[(volatile int *)x];
    x6 = 6[(volatile int *)x];
```

```
x7 =
x8 =
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x0 =
x1 =
```

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RD, LDM)
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$n$ LDR + $n$ ADD:

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```
int sum(int *x)
{
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  int x0,x1,x2,x3,x4,
      x5,x6,x7,x8,x9;

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    x0 = 0[(volatile int *)x];
    x1 = 1[(volatile int *)x];
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    x3 = 3[(volatile int *)x];
    x4 = 4[(volatile int *)x];
    x5 = 5[(volatile int *)x];
    x6 = 6[(volatile int *)x];
```

```
    x7 = 7[(vola
    x8 = 8[(vola
    x9 = 9[(vola
    result += x0
    result += x1
    result += x2
    result += x3
    result += x4
    result += x5
    result += x6
    result += x7
    result += x8
    result += x9
    x0 = 10[(vol
    x1 = 11[(vol
```

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```
int sum(int *x)
{
  int result = 0;
  int *y = x + 1000;
  int x0,x1,x2,x3,x4,
      x5,x6,x7,x8,x9;

  while (x != y) {
    x0 = 0[(volatile int *)x];
    x1 = 1[(volatile int *)x];
    x2 = 2[(volatile int *)x];
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```

```
x7 = 7[(volatile int
x8 = 8[(volatile int
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result += x0;
result += x1;
result += x2;
result += x3;
result += x4;
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result += x6;
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```

```
int sum(int *x)
{
  int result = 0;
  int *y = x + 1000;
  int x0,x1,x2,x3,x4,
      x5,x6,x7,x8,x9;

  while (x != y) {
    x0 = 0[(volatile int *)x];
    x1 = 1[(volatile int *)x];
    x2 = 2[(volatile int *)x];
    x3 = 3[(volatile int *)x];
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```



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    x7 = 7[(volatile int *)x];
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    result += x0;
    result += x1;
    result += x2;
    result += x3;
    result += x4;
    result += x5;
    result += x6;
    result += x7;
    result += x8;
    result += x9;
    x0 = 10[(volatile int *)x];
    x1 = 11[(volatile int *)x];
```

```
(int *x)

esult = 0;
y = x + 1000;
0,x1,x2,x3,x4,
5,x6,x7,x8,x9;

 (x != y) {
= 0[(volatile int *)x];
= 1[(volatile int *)x];
= 2[(volatile int *)x];
= 3[(volatile int *)x];
= 4[(volatile int *)x];
= 5[(volatile int *)x];
= 6[(volatile int *)x];
```



```
x7 = 7[(volatile int *)x];
x8 = 8[(volatile int *)x];
x9 = 9[(volatile int *)x];
result += x0;
result += x1;
result += x2;
result += x3;
result += x4;
result += x5;
result += x6;
result += x7;
result += x8;
result += x9;
x0 = 10[(volatile int *)x];
x1 = 11[(volatile int *)x];
```

```
x2 =
x3 =
x4 =
x5 =
x6 =
x7 =
x8 =
x9 =
x +=
resu
resu
resu
resu
resu
resu
```

```
;
000;
3,x4,
8,x9;

 {
tile int *)x];
tile int *)x];
tile int *)x];
tile int *)x];
tile int *)x];
tile int *)x];
tile int *)x];
```



```
x7 = 7[(volatile int *)x];
x8 = 8[(volatile int *)x];
x9 = 9[(volatile int *)x];
result += x0;
result += x1;
result += x2;
result += x3;
result += x4;
result += x5;
result += x6;
result += x7;
result += x8;
result += x9;
x0 = 10[(volatile int *)x];
x1 = 11[(volatile int *)x];
```

```
x2 = 12[(vol
x3 = 13[(vol
x4 = 14[(vol
x5 = 15[(vol
x6 = 16[(vol
x7 = 17[(vol
x8 = 18[(vol
x9 = 19[(vol
x += 20;
result += x0
result += x1
result += x2
result += x3
result += x4
result += x5
```

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```
*)x];
*)x];
*)x];
*)x];
*)x];
*)x];
*)x];
```

```
x7 = 7[(volatile int *)x];
x8 = 8[(volatile int *)x];
x9 = 9[(volatile int *)x];
result += x0;
result += x1;
result += x2;
result += x3;
result += x4;
result += x5;
result += x6;
result += x7;
result += x8;
result += x9;
x0 = 10[(volatile int *)x];
x1 = 11[(volatile int *)x];
```

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```
x2 = 12[(volatile int
x3 = 13[(volatile int
x4 = 14[(volatile int
x5 = 15[(volatile int
x6 = 16[(volatile int
x7 = 17[(volatile int
x8 = 18[(volatile int
x9 = 19[(volatile int
x += 20;
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result += x9;
x0 = 10[(volatile int *)x];
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```



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x6 = 16[(volatile int *)x];
x7 = 17[(volatile int *)x];
x8 = 18[(volatile int *)x];
x9 = 19[(volatile int *)x];
x += 20;
result += x0;
result += x1;
result += x2;
result += x3;
result += x4;
result += x5;
```

```
= 7[(volatile int *)x];
= 8[(volatile int *)x];
= 9[(volatile int *)x];
ult += x0;
ult += x1;
ult += x2;
ult += x3;
ult += x4;
ult += x5;
ult += x6;
ult += x7;
ult += x8;
ult += x9;
= 10[(volatile int *)x];
= 11[(volatile int *)x];
```



```
x2 = 12[(volatile int *)x];
x3 = 13[(volatile int *)x];
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x8 = 18[(volatile int *)x];
x9 = 19[(volatile int *)x];
x += 20;
result += x0;
result += x1;
result += x2;
result += x3;
result += x4;
result += x5;
```

```
    resu
    resu
    resu
    resu
  }

  returr
}
```

```
tile int *)x];
tile int *)x];
tile int *)x];
;
;
;
;
;
;
;
;
;
;
atile int *)x];
atile int *)x];
```



```
x2 = 12[(volatile int *)x];
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x4 = 14[(volatile int *)x];
x5 = 15[(volatile int *)x];
x6 = 16[(volatile int *)x];
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x += 20;
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result += x2;
result += x3;
result += x4;
result += x5;
```

```
    result += x6
    result += x7
    result += x8
    result += x9
  }

  return result;
}
```

```
*)x];
*)x];
*)x];
. *)x];
. *)x];
```



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  x2 = 12[(volatile int *)x];
  x3 = 13[(volatile int *)x];
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  x5 = 15[(volatile int *)x];
  x6 = 16[(volatile int *)x];
  x7 = 17[(volatile int *)x];
  x8 = 18[(volatile int *)x];
  x9 = 19[(volatile int *)x];
  x += 20;
  result += x0;
  result += x1;
  result += x2;
  result += x3;
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    result += x6;
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result += x4;
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    result += x8;
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result += x3;
result += x4;
result += x5;
```



```
    result += x6;
    result += x7;
    result += x8;
    result += x9;
  }

  return result;
}
```

2526 cycles. Even better in asm.

```
x2 = 12[(volatile int *)x];
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x += 20;
result += x0;
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result += x2;
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result += x4;
result += x5;
```



```
    result += x6;
    result += x7;
    result += x8;
    result += x9;
  }

  return result;
}
```

2526 cycles. Even better in asm.

Wikipedia: "By the late 1990s for even performance sensitive code, optimizing compilers exceeded the performance of human experts."

```
x2 = 12[(volatile int *)x];
x3 = 13[(volatile int *)x];
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x5 = 15[(volatile int *)x];
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result += x0;
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result += x3;
result += x4;
result += x5;
```



```
    result += x6;
    result += x7;
    result += x8;
    result += x9;
  }

  return result;
}
```

2526 cycles. Even better in asm.

Wikipedia: "By the late 1990s for even performance sensitive code, optimizing compilers exceeded the performance of human experts."
— [citation needed]

```
= 12[(volatile int *)x];
= 13[(volatile int *)x];
= 14[(volatile int *)x];
= 15[(volatile int *)x];
= 16[(volatile int *)x];
= 17[(volatile int *)x];
= 18[(volatile int *)x];
= 19[(volatile int *)x];
= 20;
ult += x0;
ult += x1;
ult += x2;
ult += x3;
ult += x4;
ult += x5;
```



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    result += x6;
    result += x7;
    result += x8;
    result += x9;
  }

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Salsa20 reference
30.25 cycles/byte

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```
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$20 \cdot 16$ 1-cycle XORs,
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ARMv7-M instruction set
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Detailed benchmarks show
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CPUs are evolving
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## The big picture

CPUs are evolving farther and farther away from naive models of CPUs.

Minor optimization challenges:

- Pipelining.
- Superscalar processing.

Major optimization challenges:

- Vectorization.
- Many threads; many cores.
- The memory hierarchy; the ring; the mesh.
- Larger-scale parallelism.
- Larger-scale networking.

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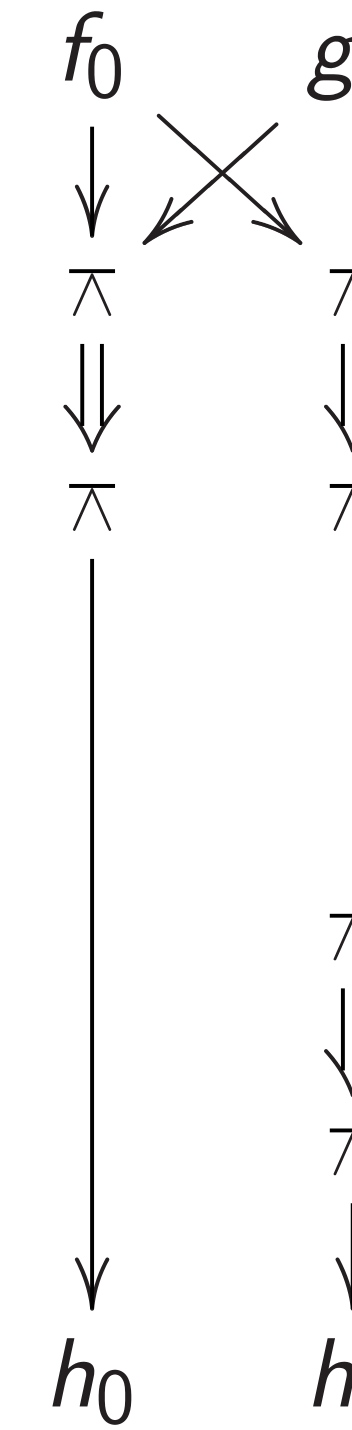


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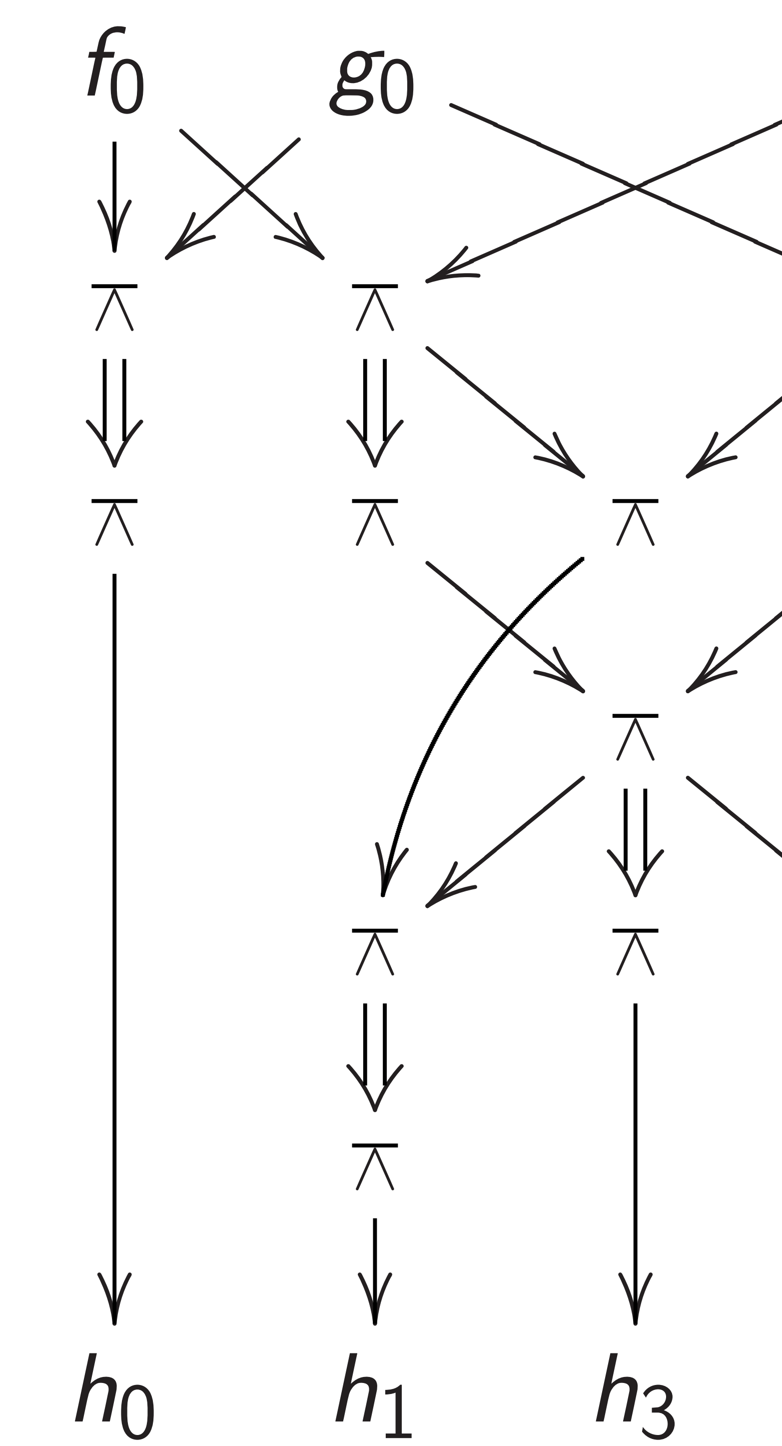


Gates $\barwedge : a, b \mapsto 1$
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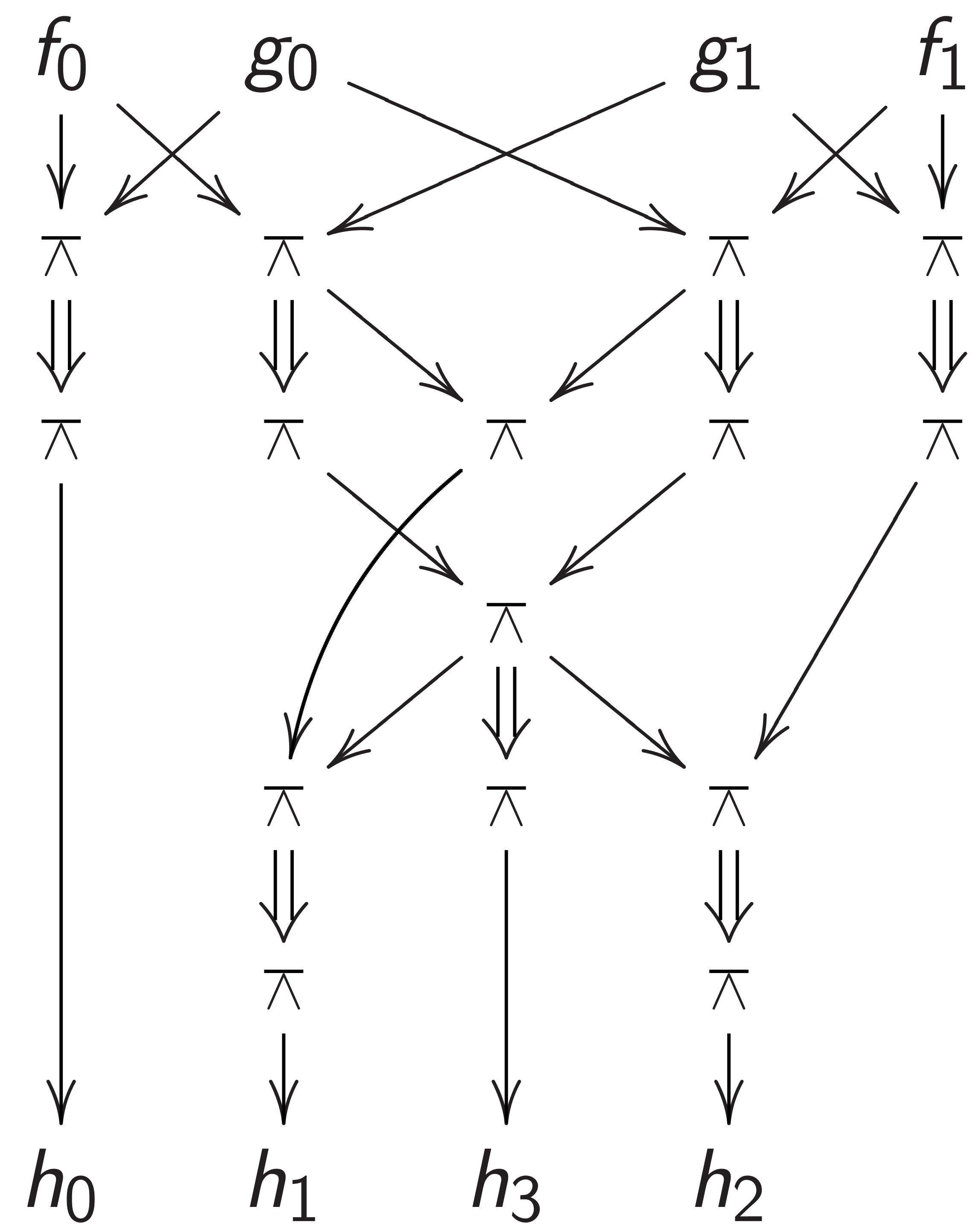
## CPU design in a nutshell




Gates $\barwedge : a, b \mapsto 1 - ab$ com
product $h_0 + 2h_1 + 4h_2 + 8$
of integers $f_0 + 2f_1, g_0 + 2g$

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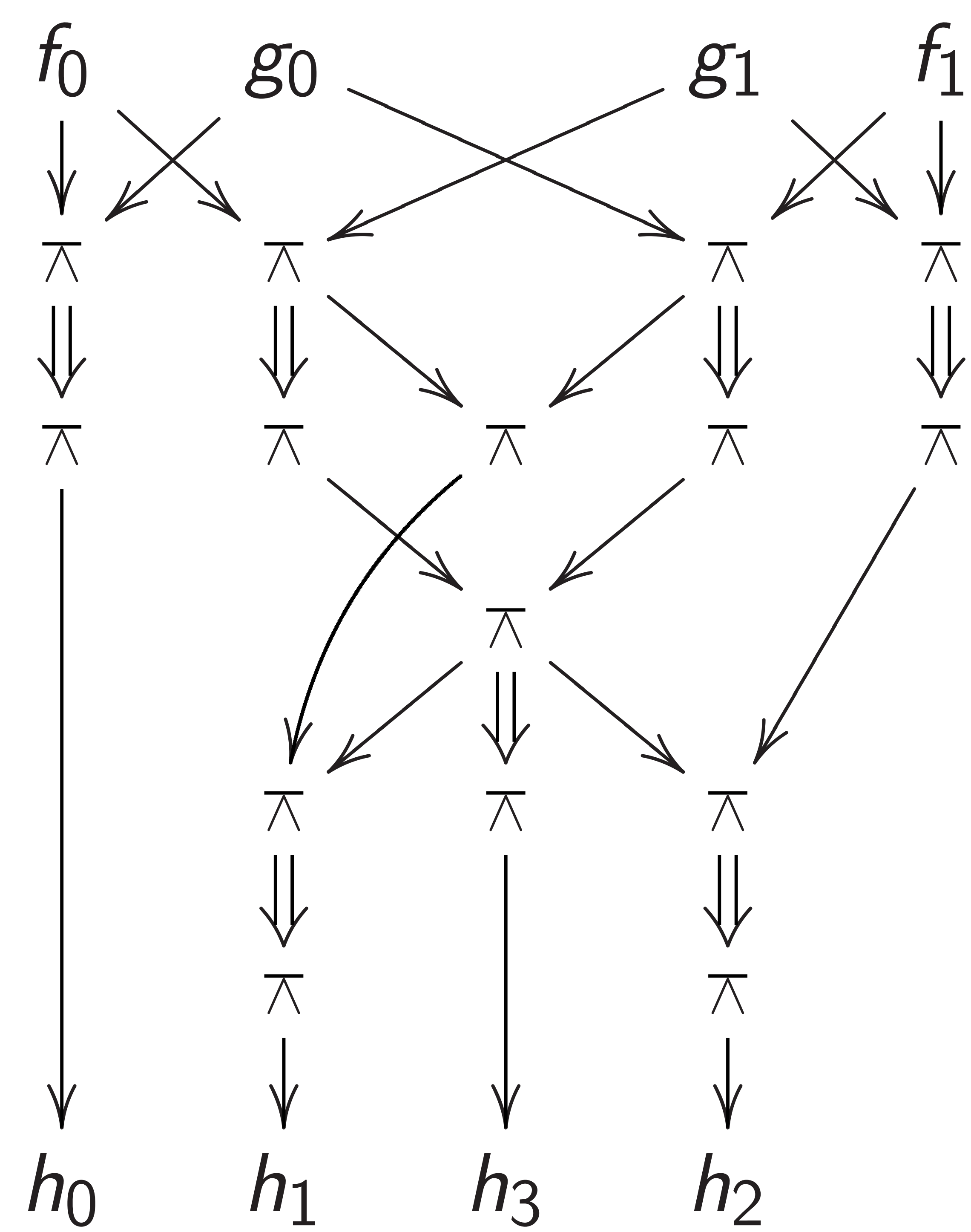


Gates $\barwedge : a, b \mapsto 1 - ab$ computing
product $h_0 + 2h_1 + 4h_2 + 8h_3$
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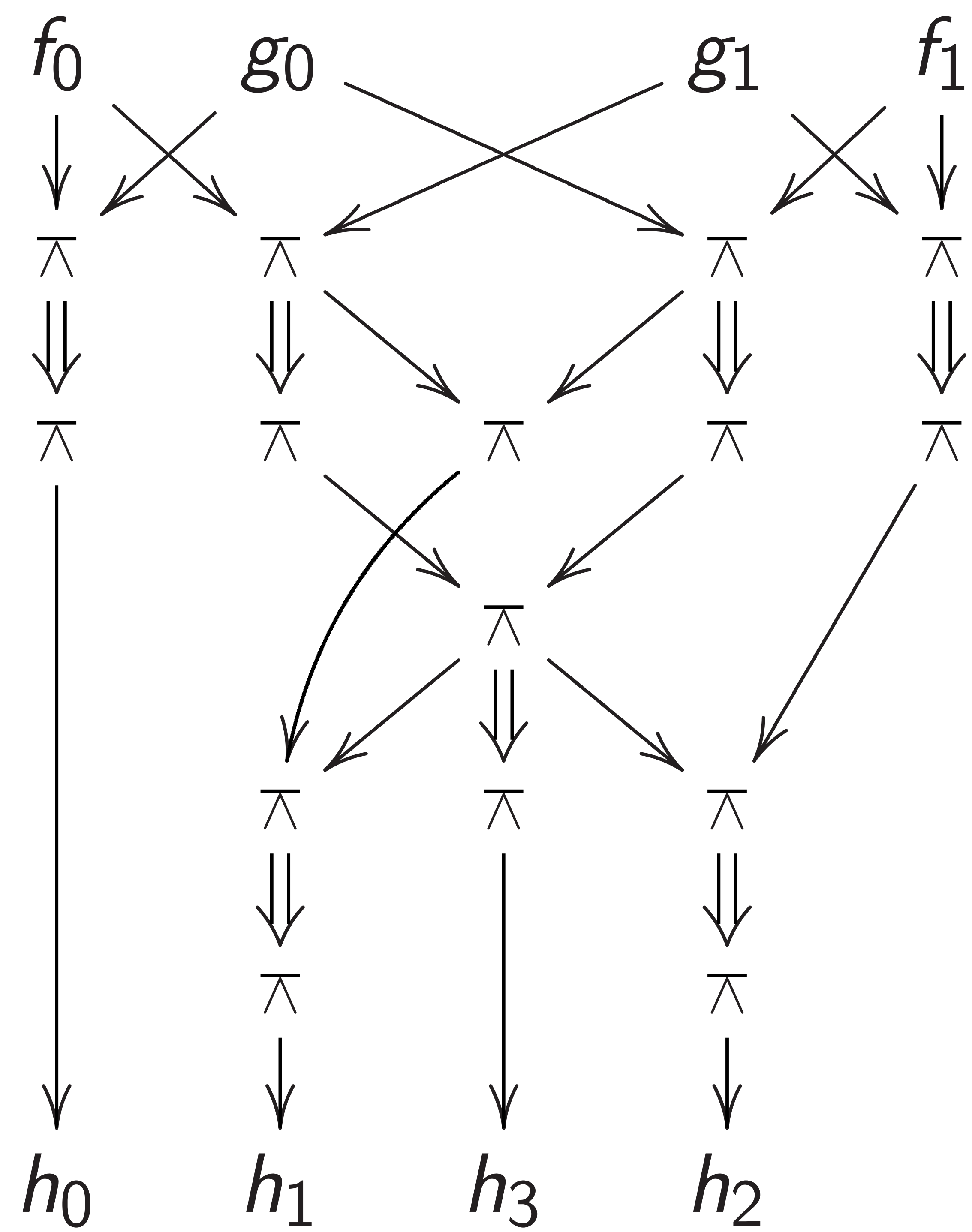


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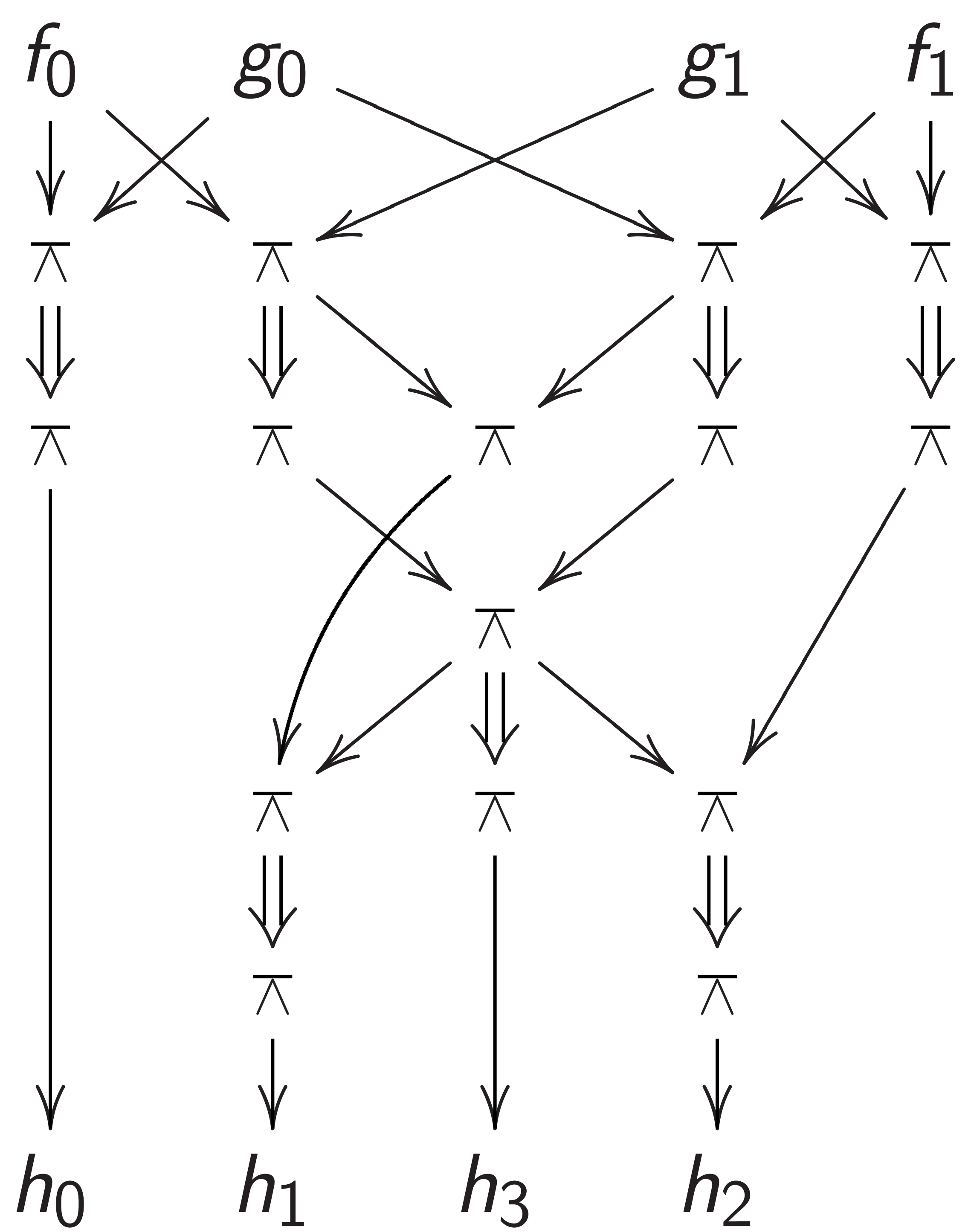


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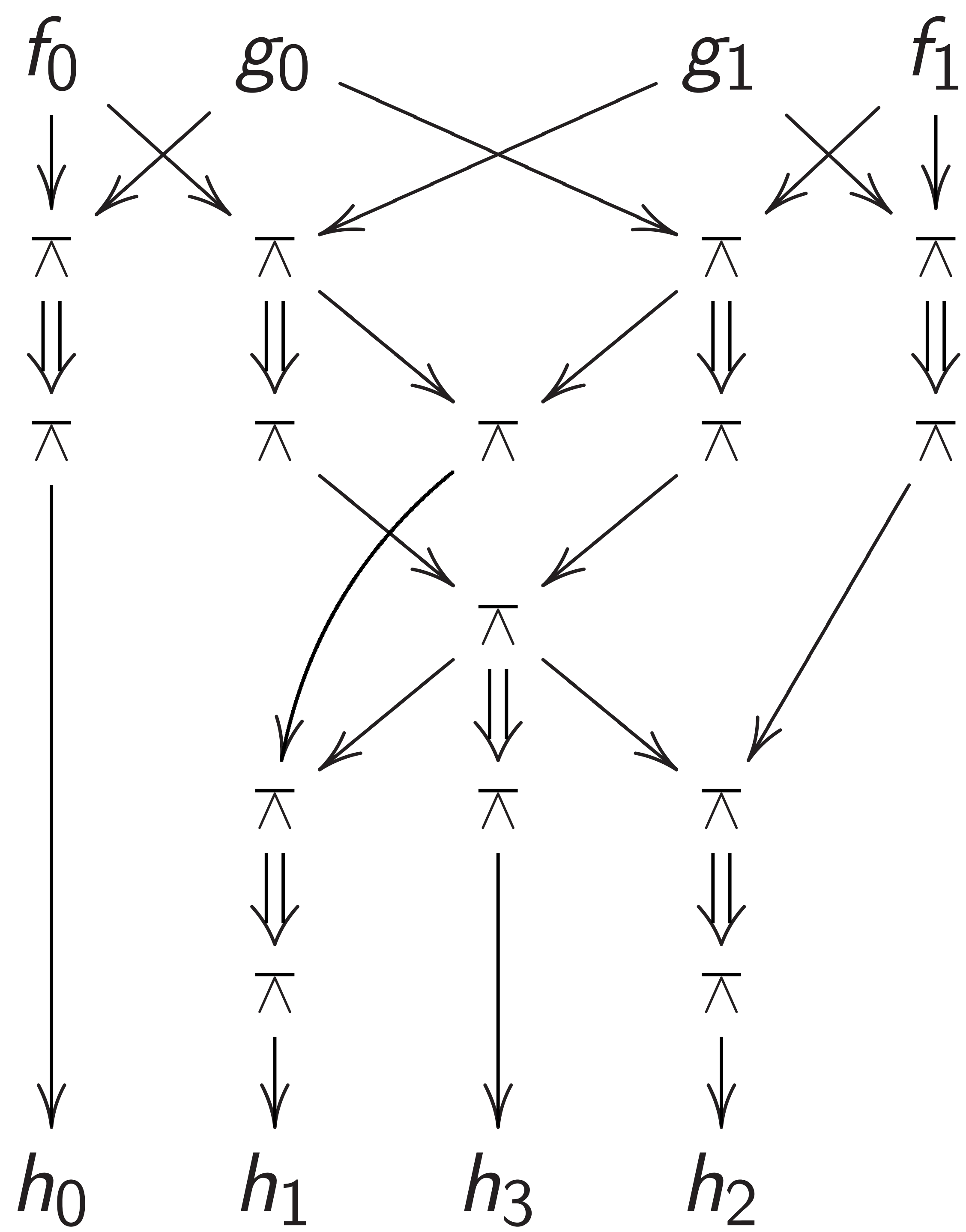


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Electricity takes time to percolate through wires and If $f_0, f_1, g_0, g_1$ are stable then $h_0, h_1, h_2, h_3$ are stable a few moments later.

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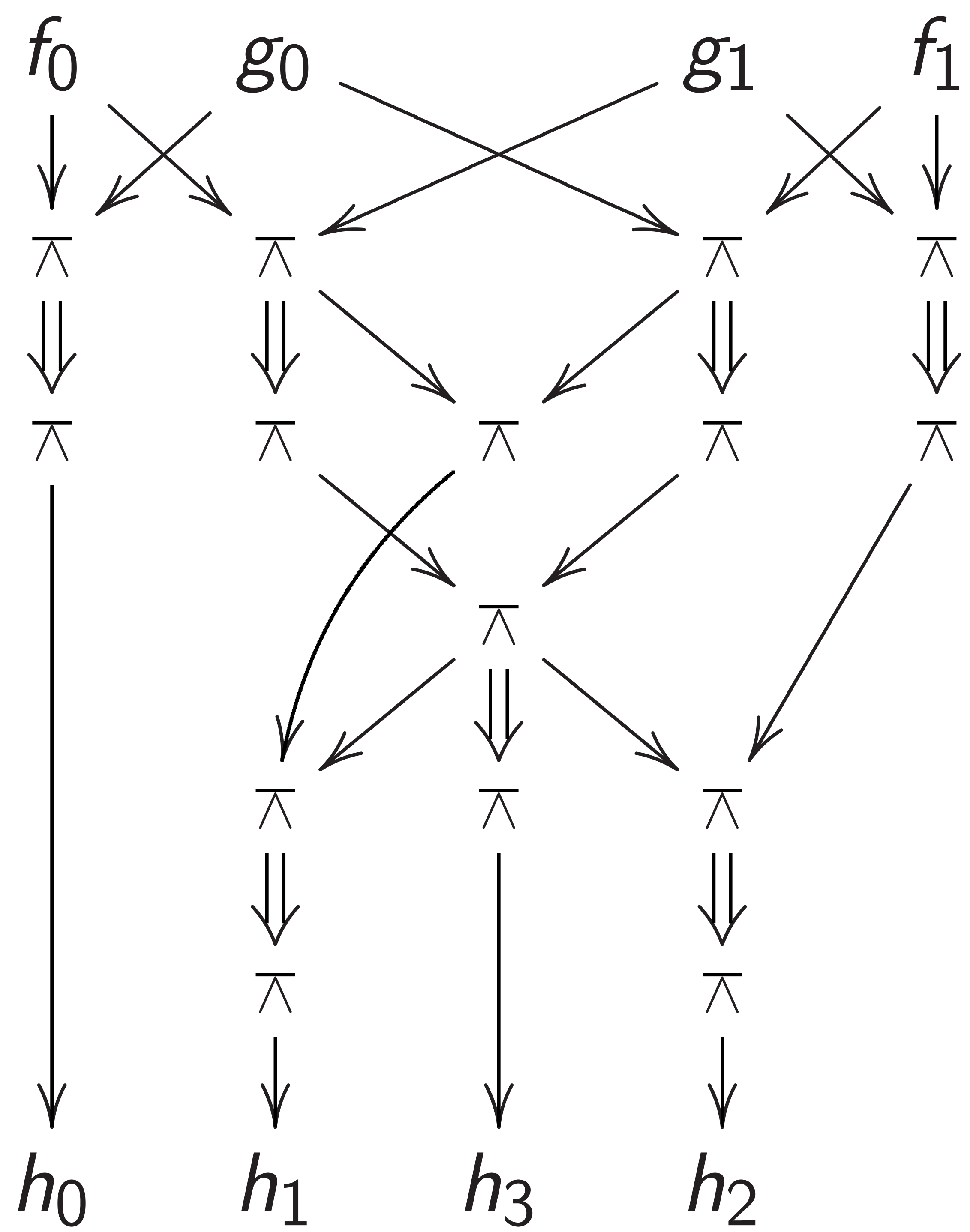


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Electricity takes time to percolate through wires and gates. If $f_0, f_1, g_0, g_1$ are stable then $h_0, h_1, h_2, h_3$ are stable a few moments later.

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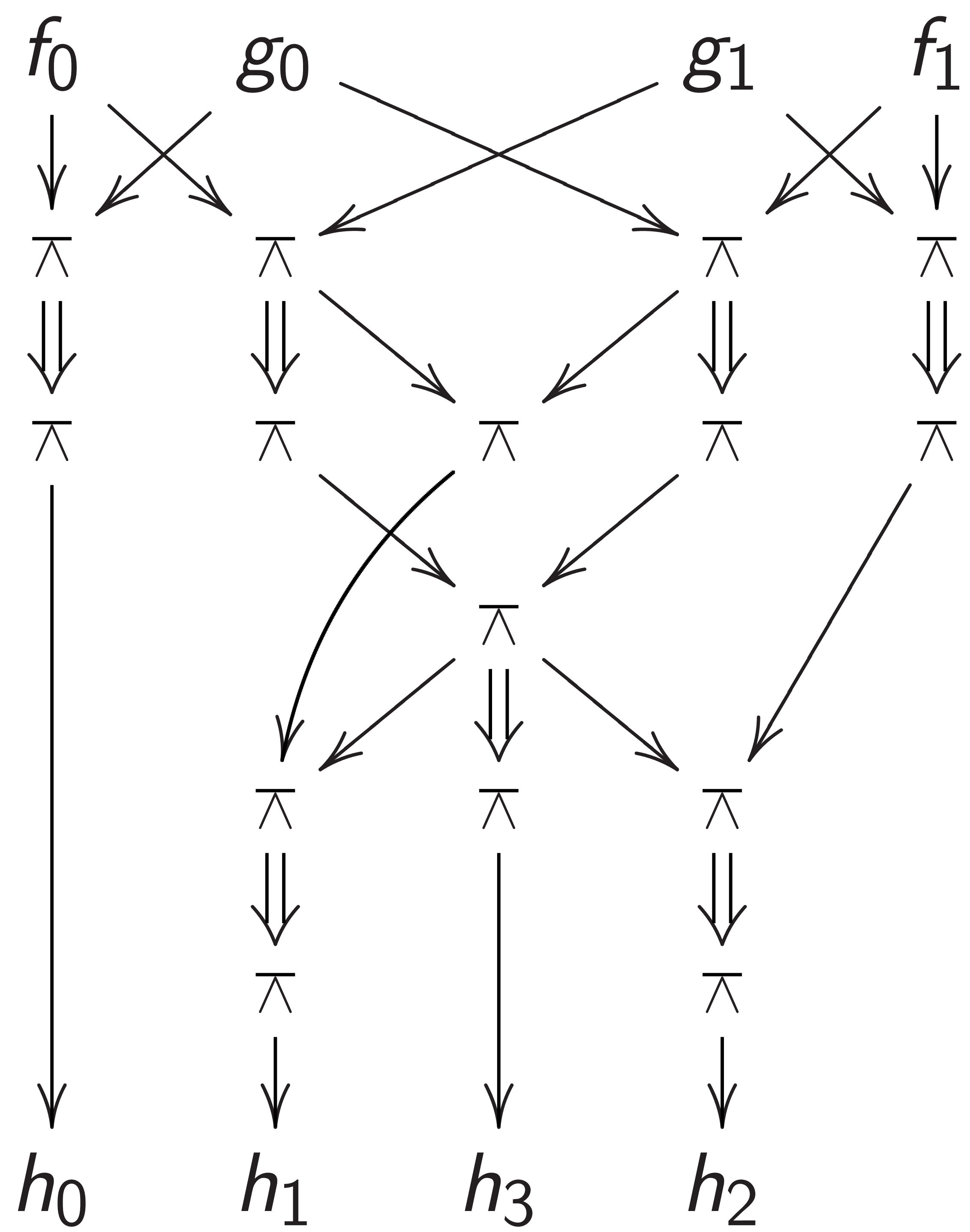


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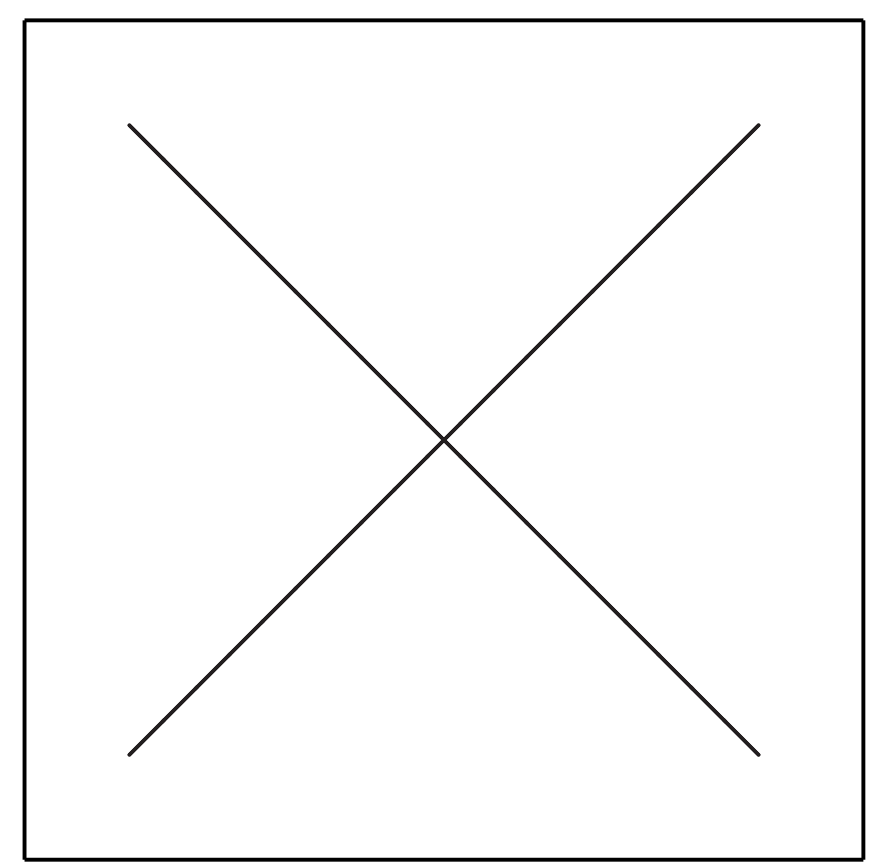
Build circuit with more gates to multiply (e.g.) 32-bit integers:



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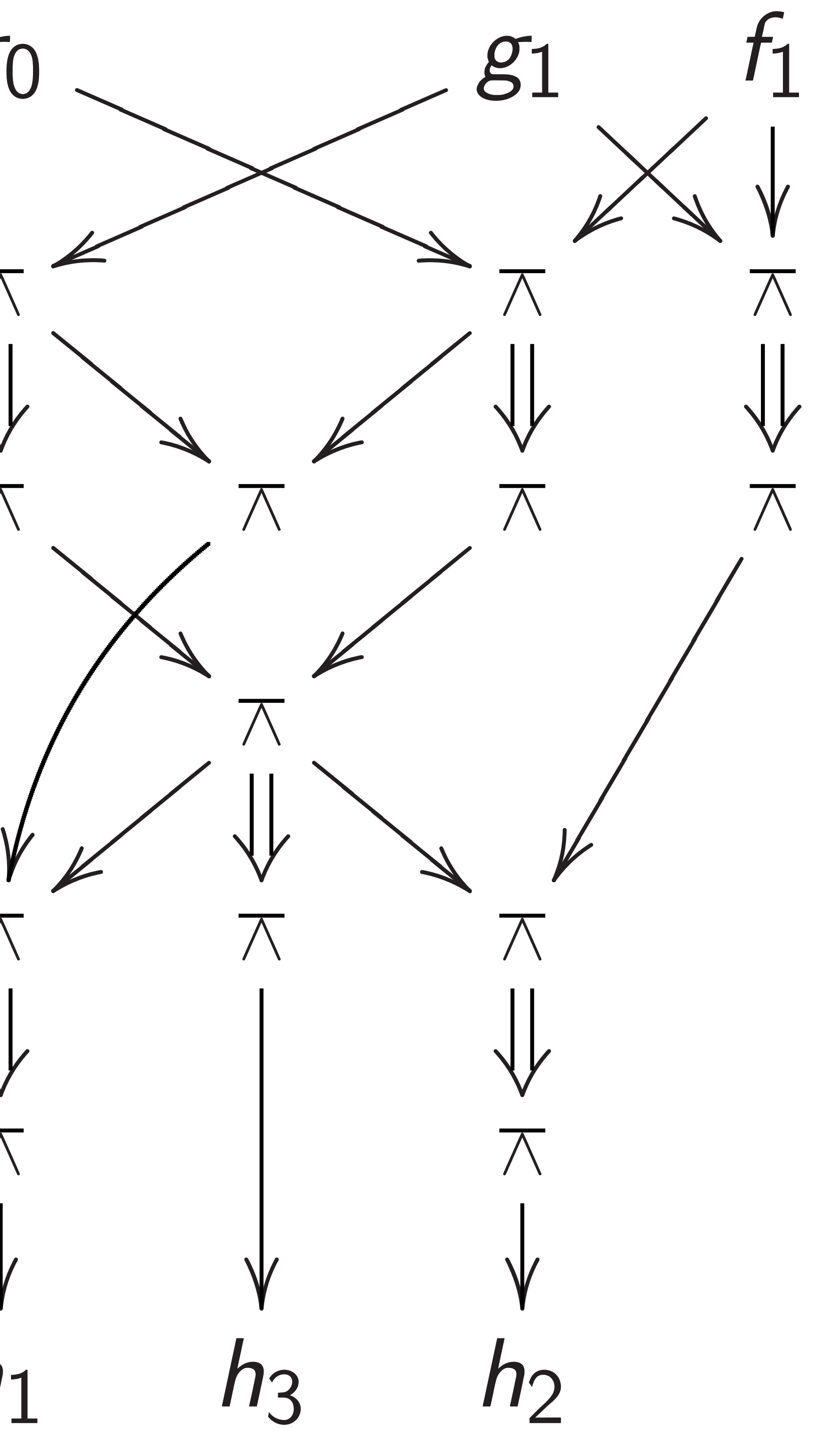


: $a, b \mapsto 1 - ab$ computing
$h_0 + 2h_1 + 4h_2 + 8h_3$
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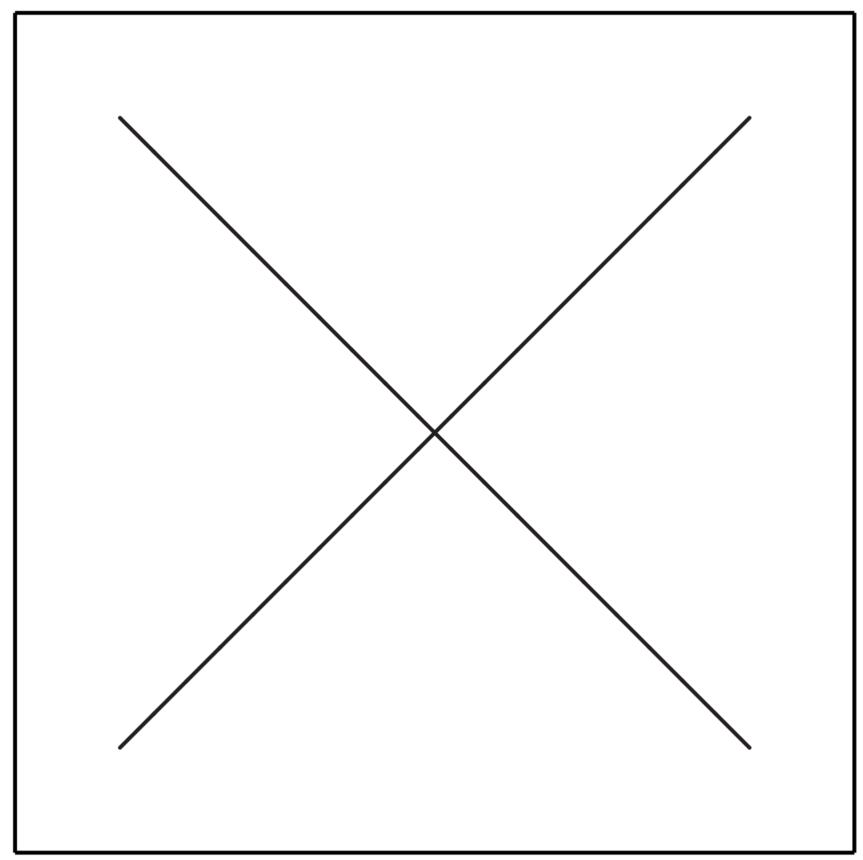

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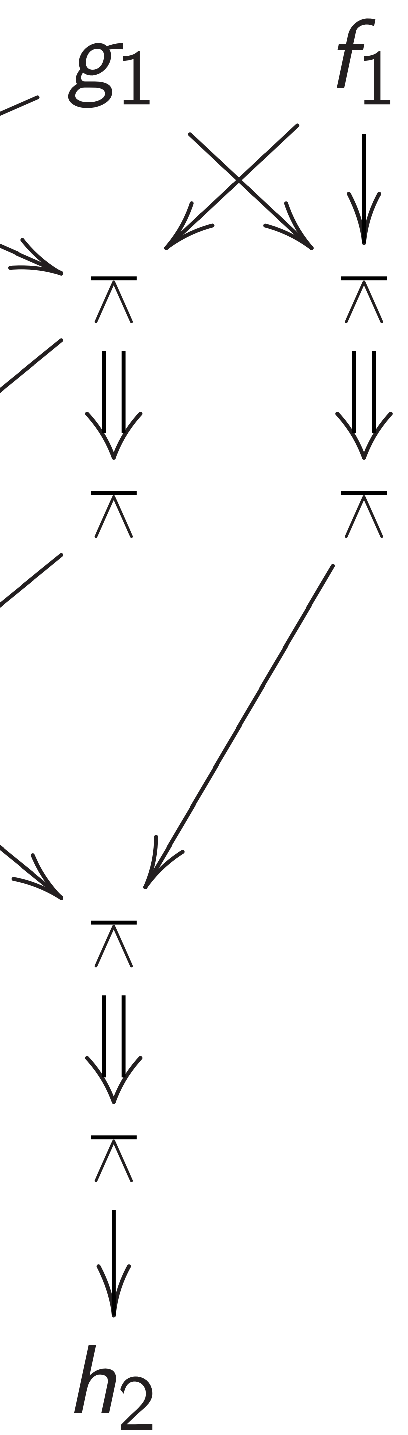


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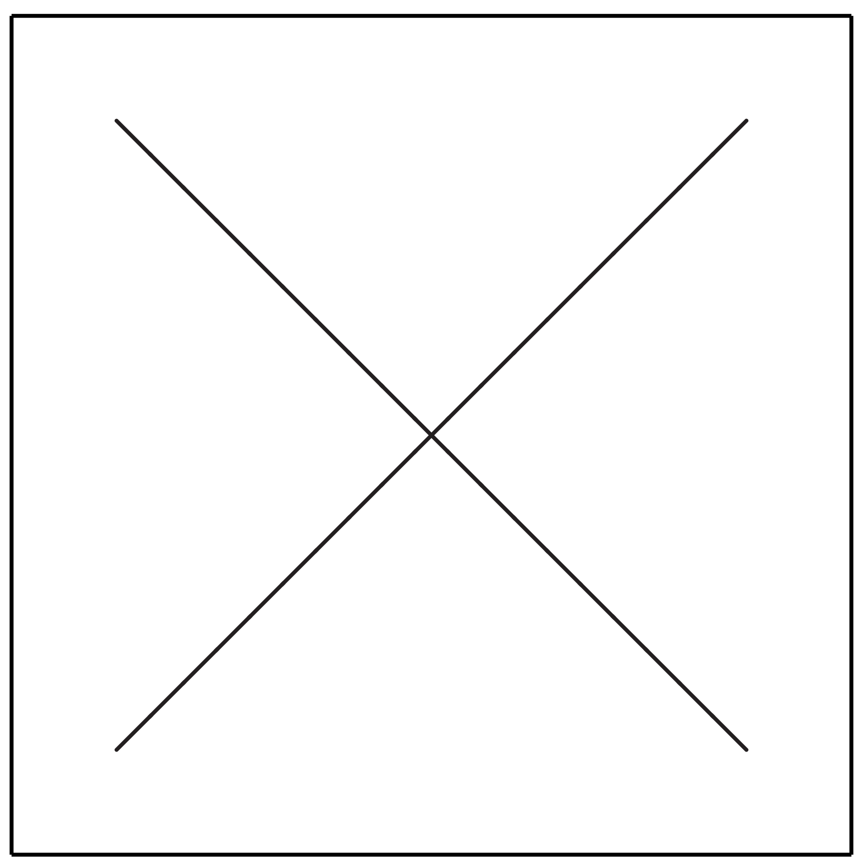

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Build circuit to co

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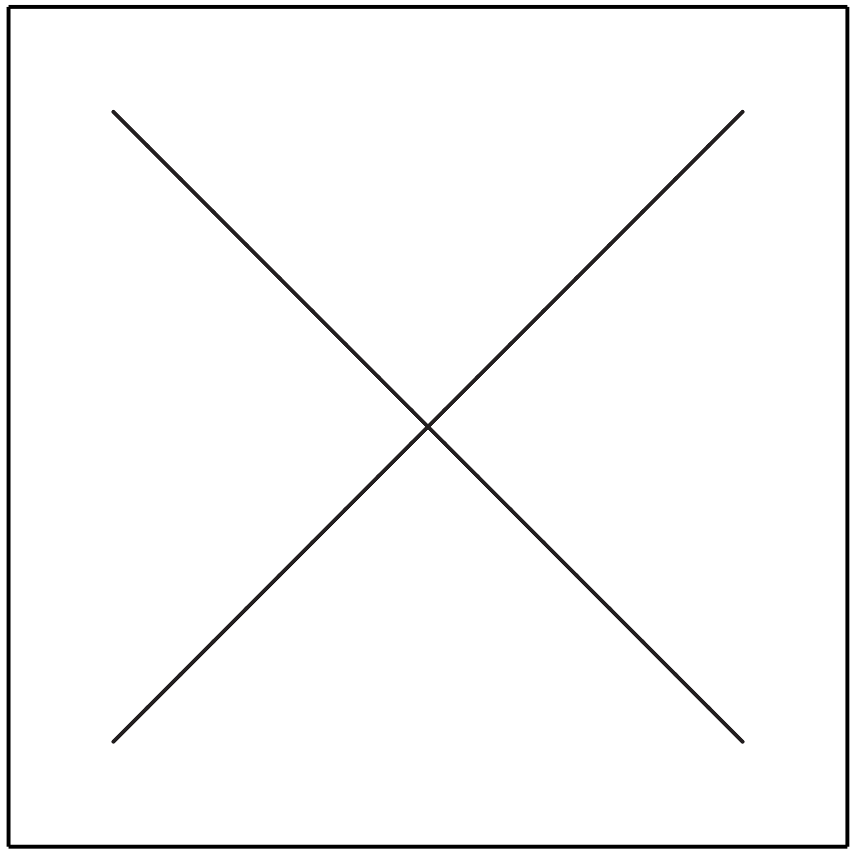

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Build circuit to compute 32-bit integer $r_i$ given 4-bit integer $i$ and 32-bit integers $r_0, r_1, \ldots$

register read

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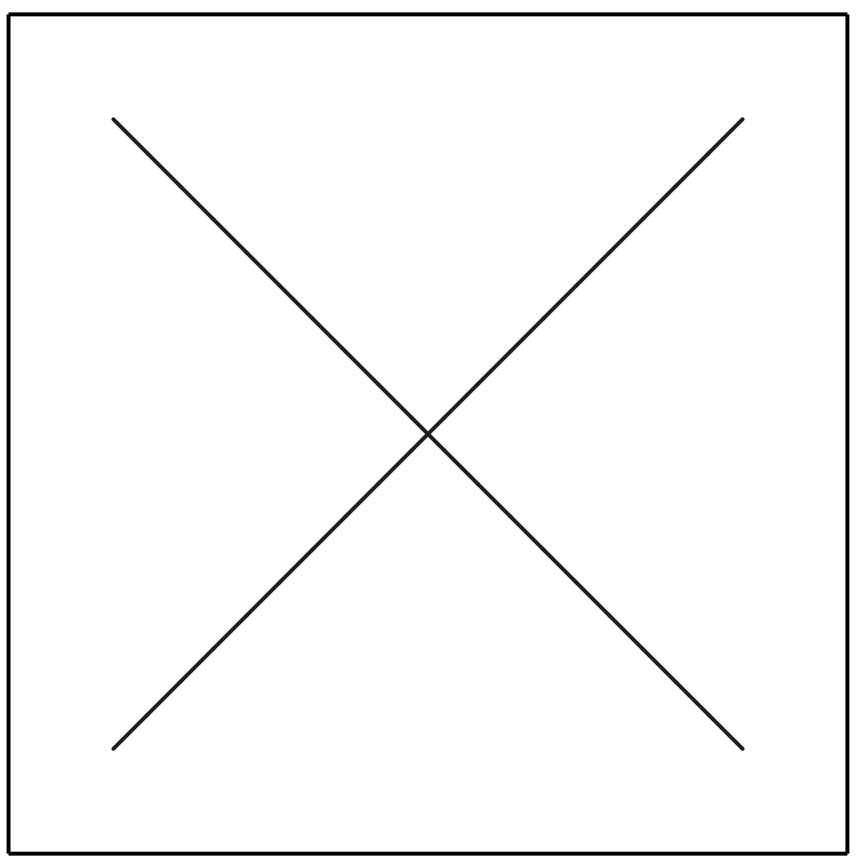

(Details omitted.)



Build circuit to compute 32-bit integer $r_i$ given 4-bit integer $i$ and 32-bit integers $r_0, r_1, \ldots, r_{15}$:

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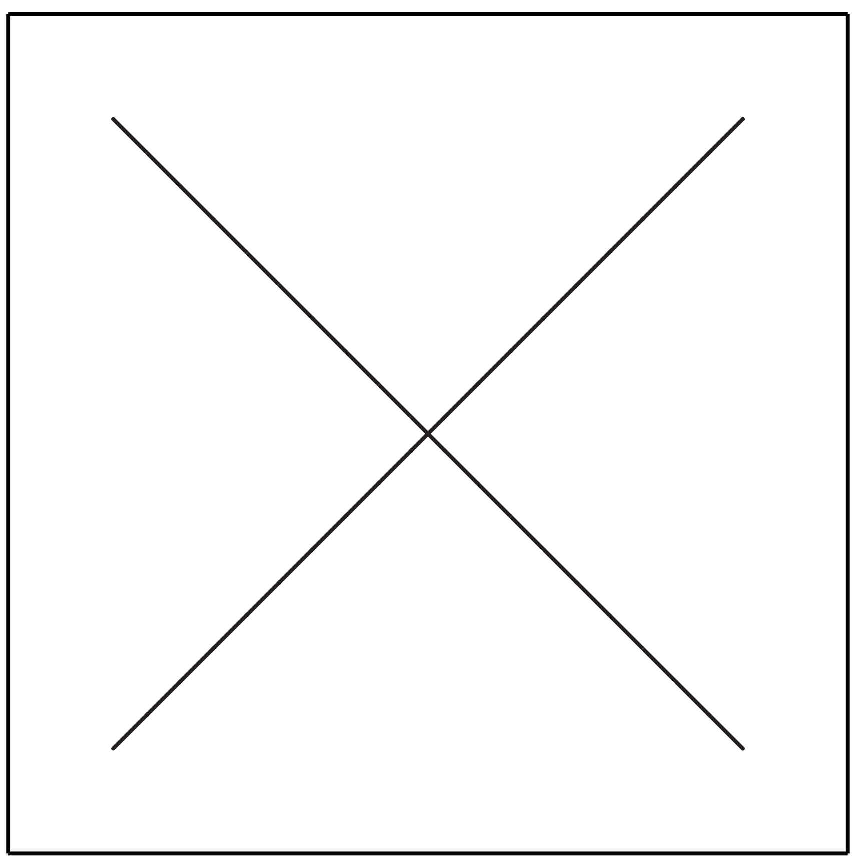


(Details omitted.)



Build circuit to compute 32-bit integer $r_i$ given 4-bit integer $i$ and 32-bit integers $r_0, r_1, \ldots, r_{15}$:

register
read

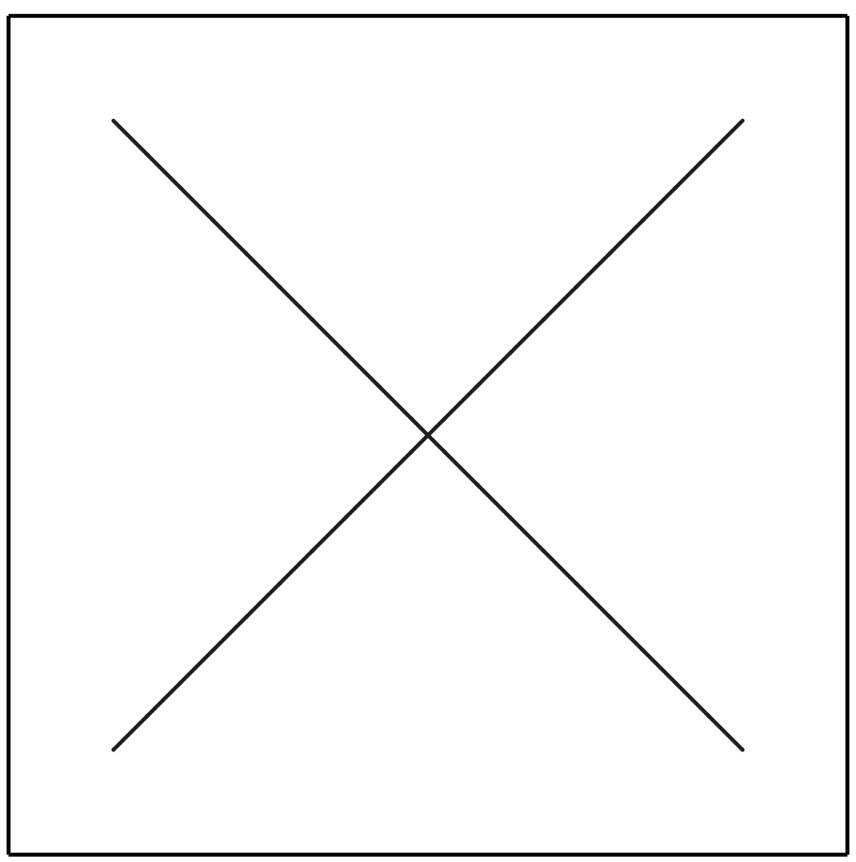
Build circuit for "register write":

$r_0, \ldots, r_{15}, s, i \mapsto r'_0, \ldots, r'_{15}$

where $r'_j = r_j$ except $r'_i = s$.

Electricity takes time to percolate through wires and gates. If $f_0, f_1, g_0, g_1$ are stable then $h_0, h_1, h_2, h_3$ are stable a few moments later.

Build circuit with more gates to multiply (e.g.) 32-bit integers:



(Details omitted.)



Build circuit to compute 32-bit integer $r_i$ given 4-bit integer $i$ and 32-bit integers $r_0, r_1, \ldots, r_{15}$:


Build circuit for "register write": $r_0, \ldots, r_{15}, s, i \mapsto r'_0, \ldots, r'_{15}$ where $r'_j = r_j$ except $r'_i = s$. Build circuit for addition. Etc.

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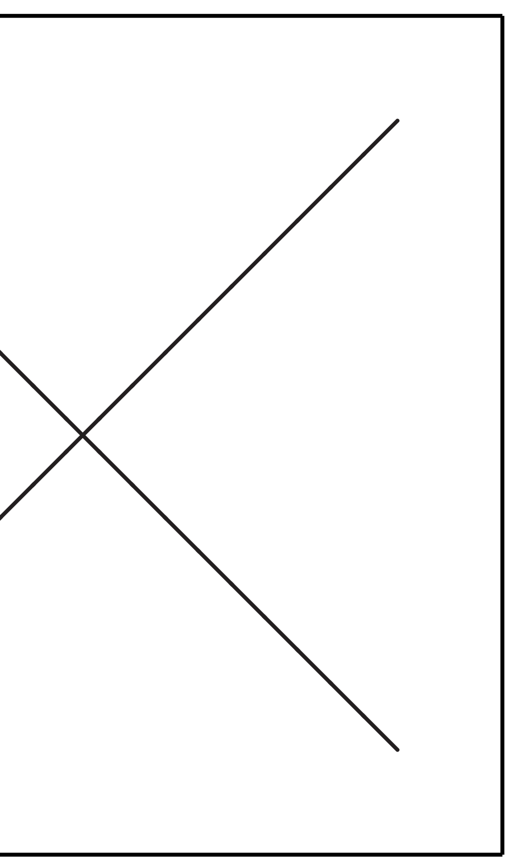

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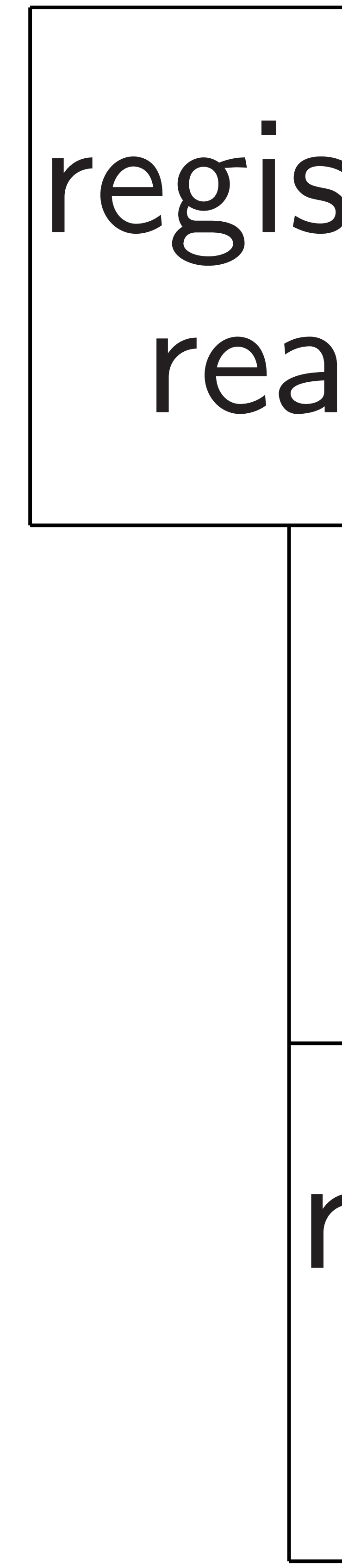

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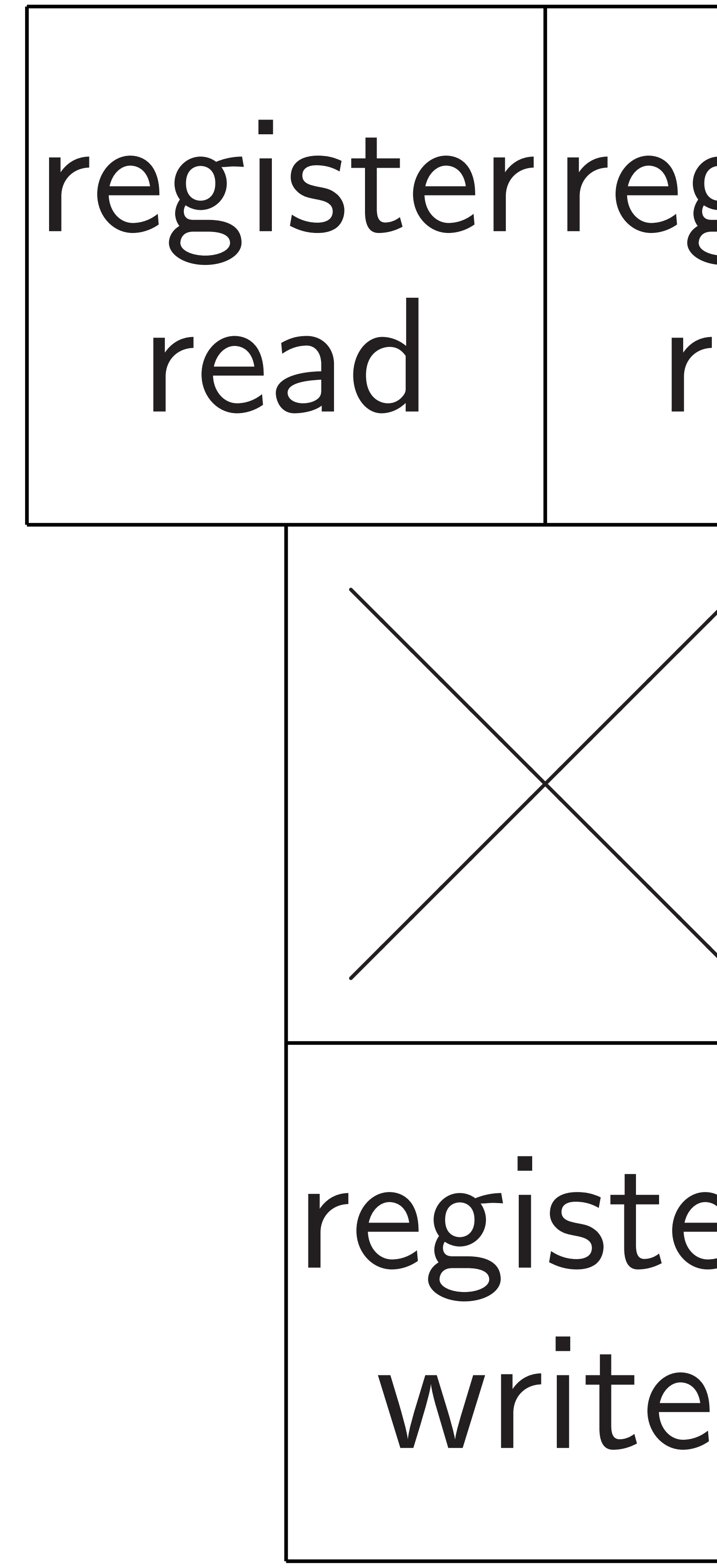

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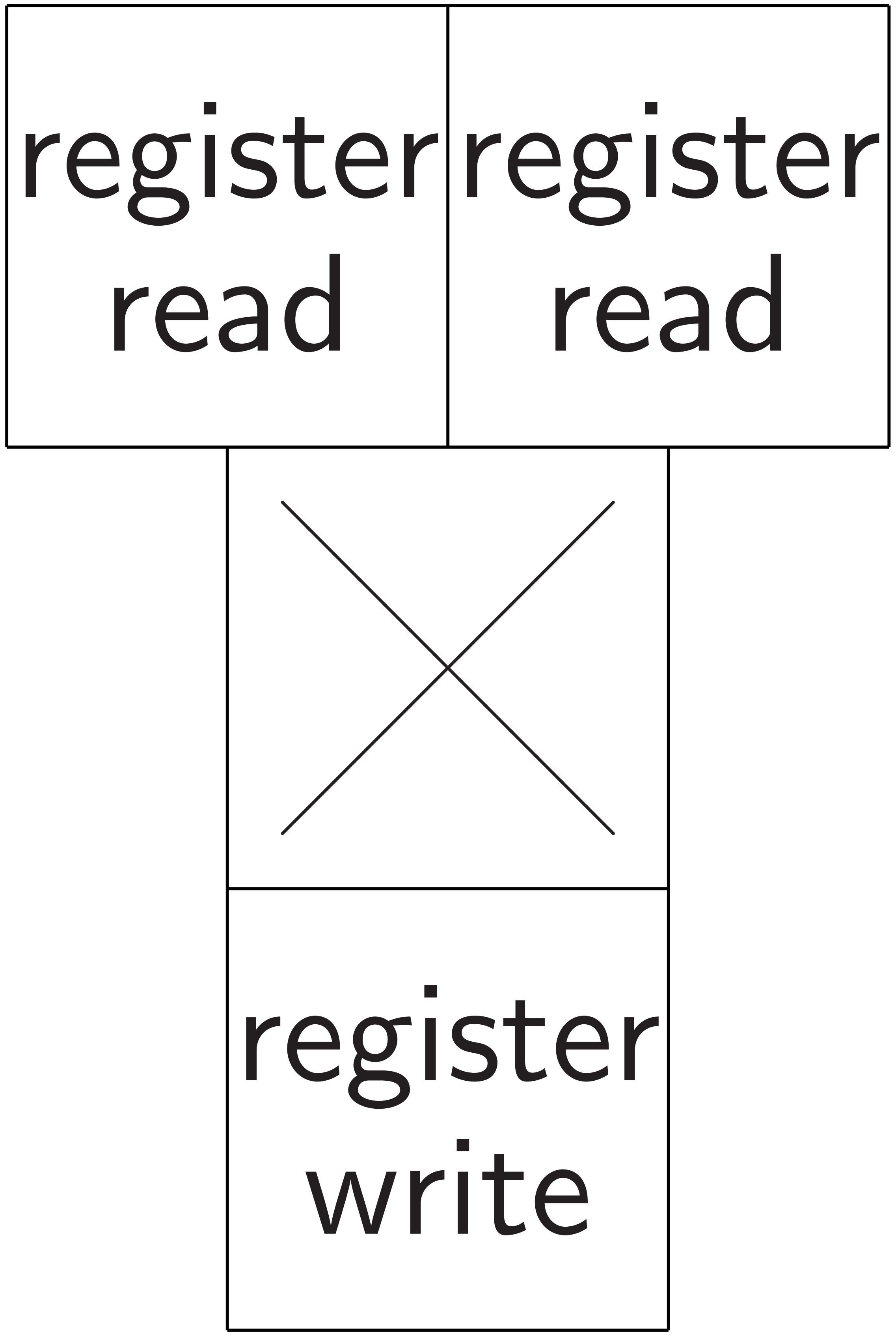

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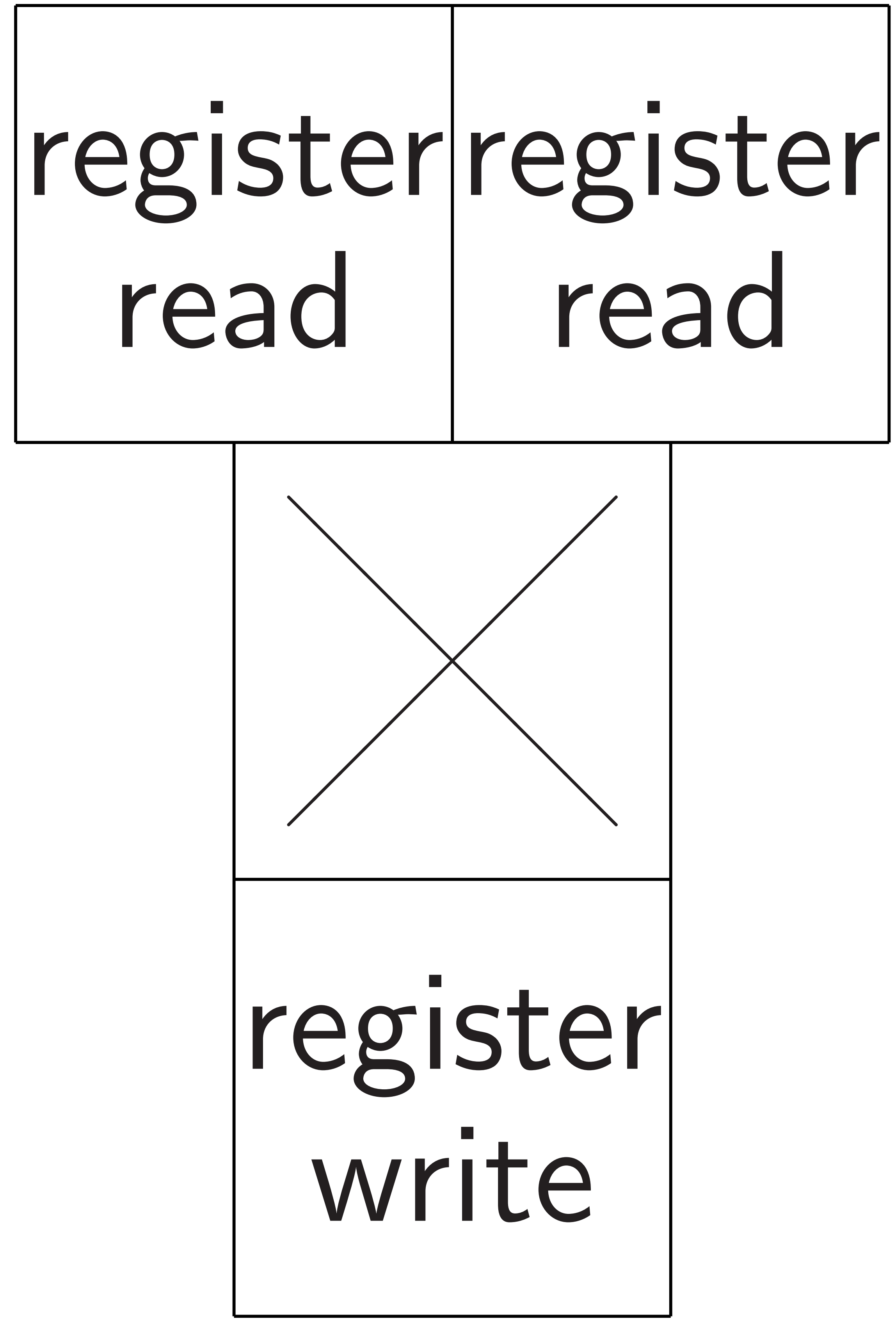

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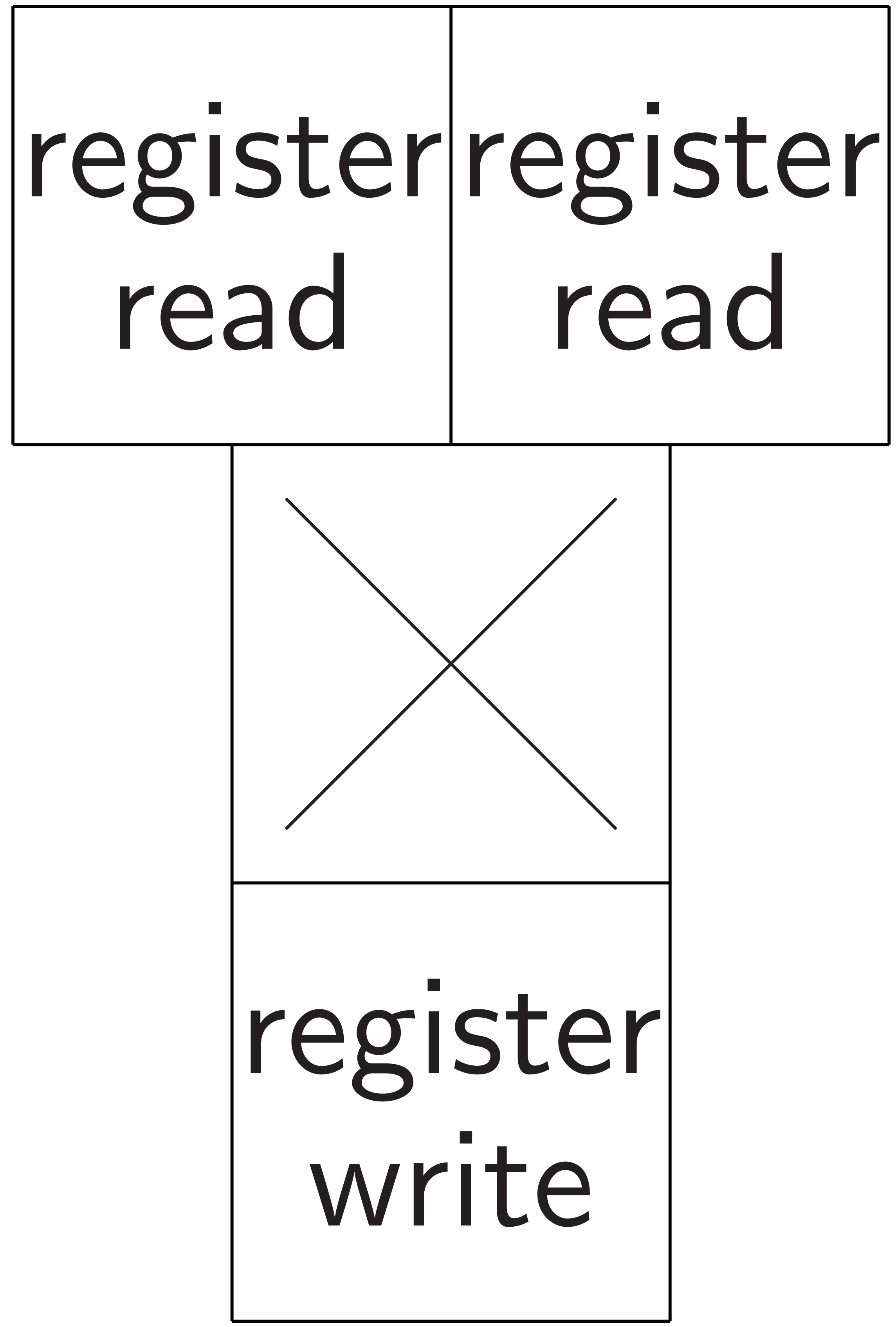


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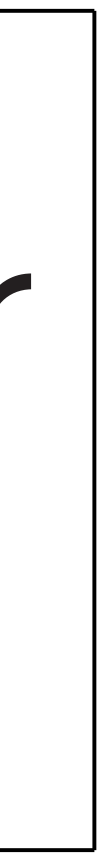

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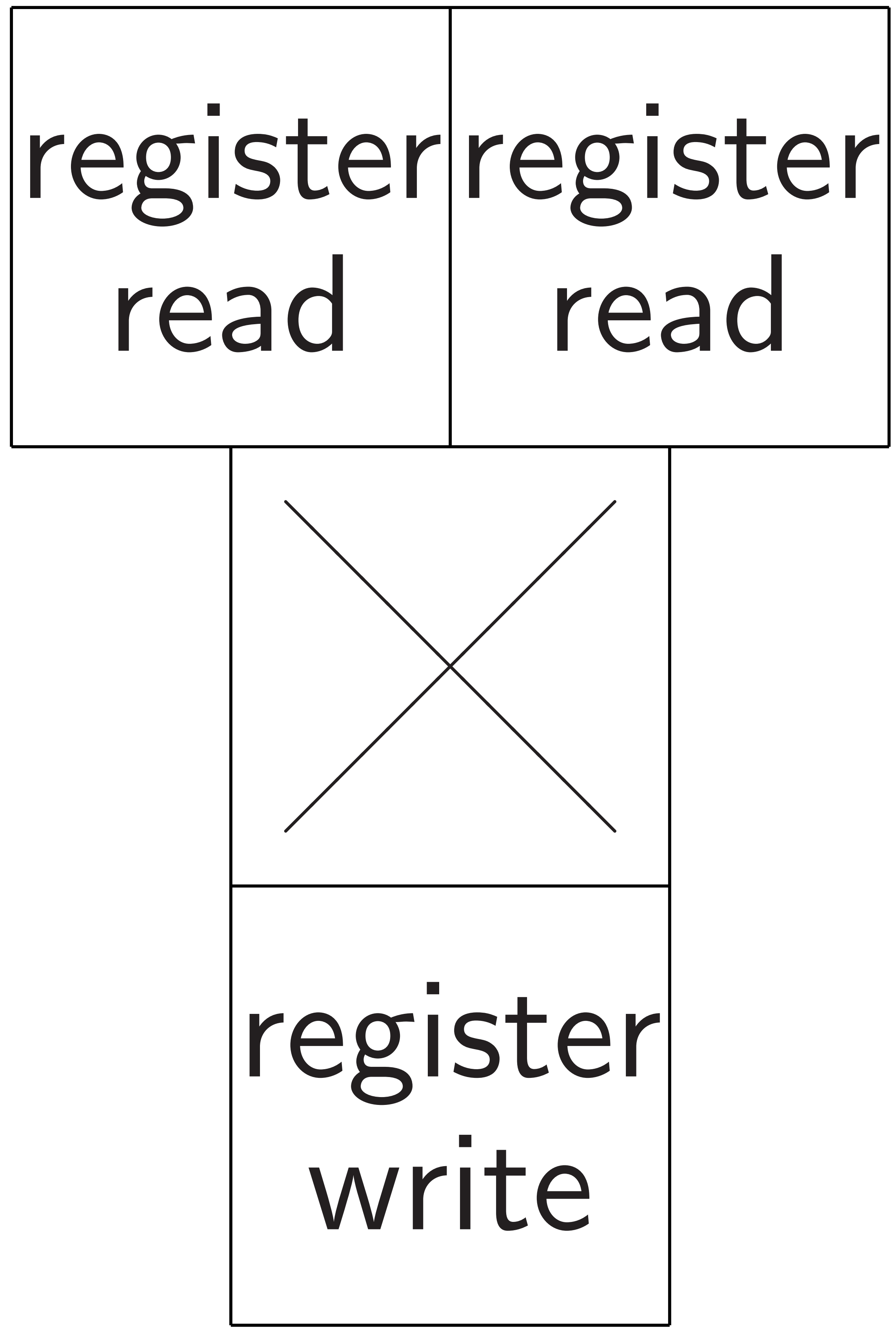


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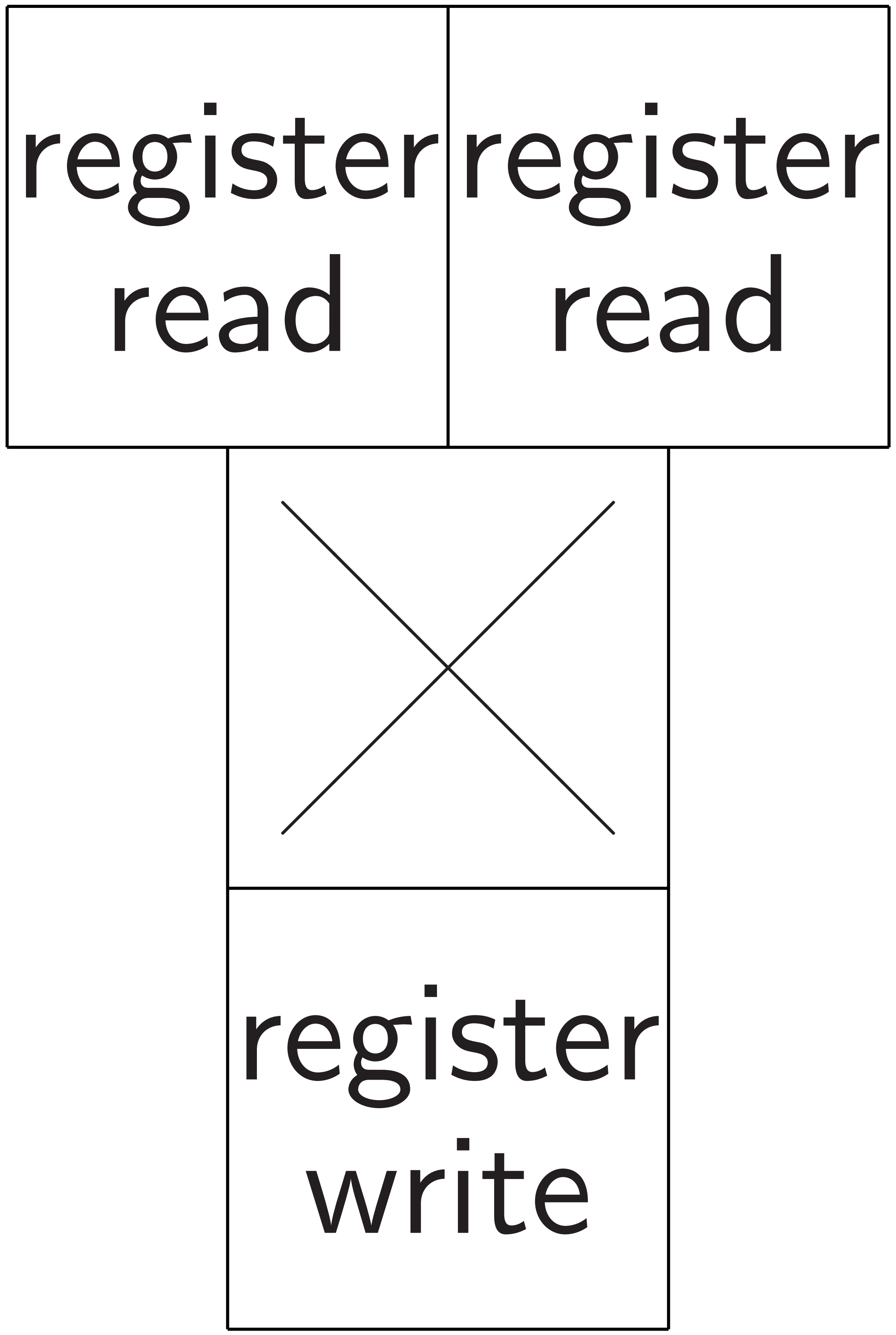




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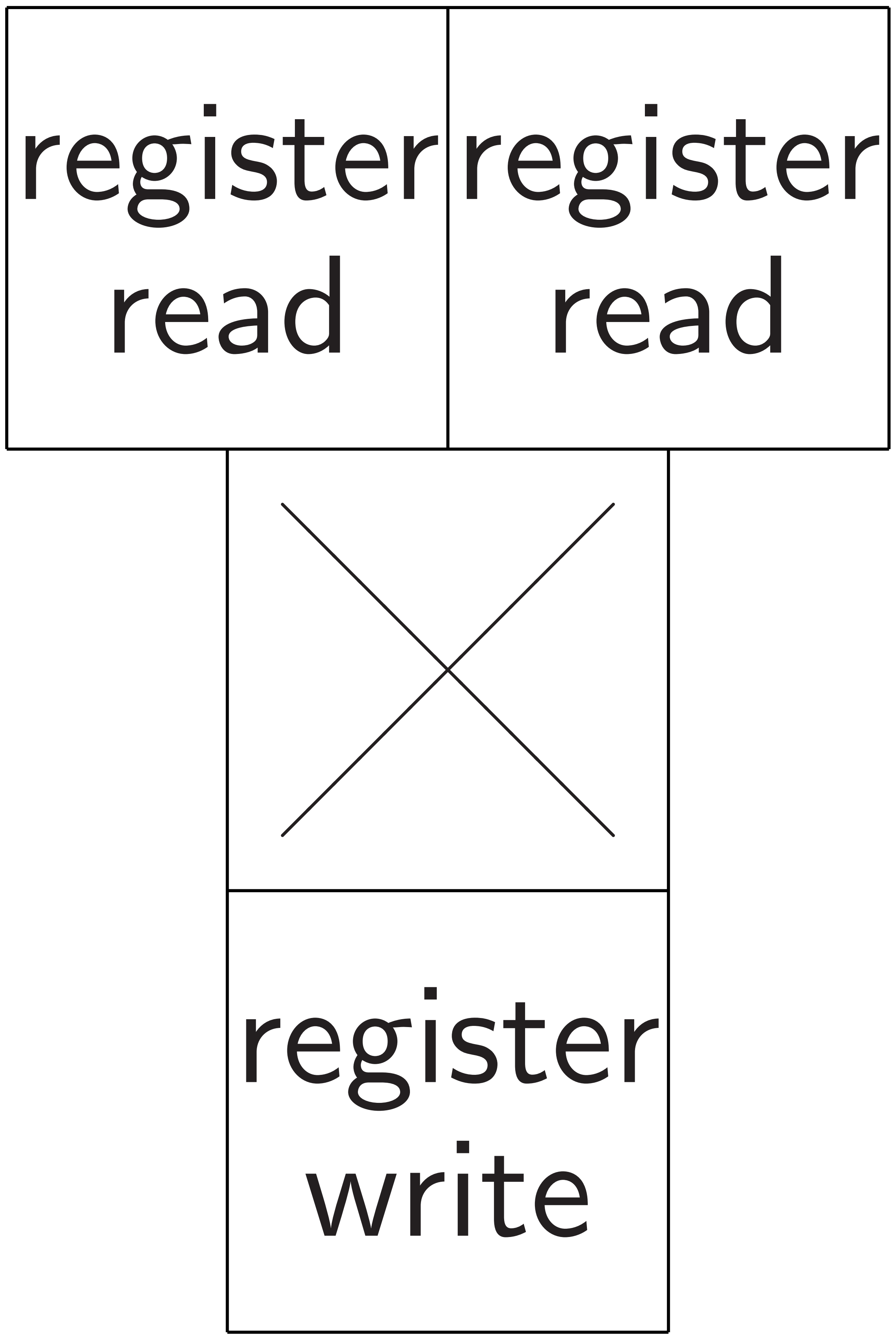




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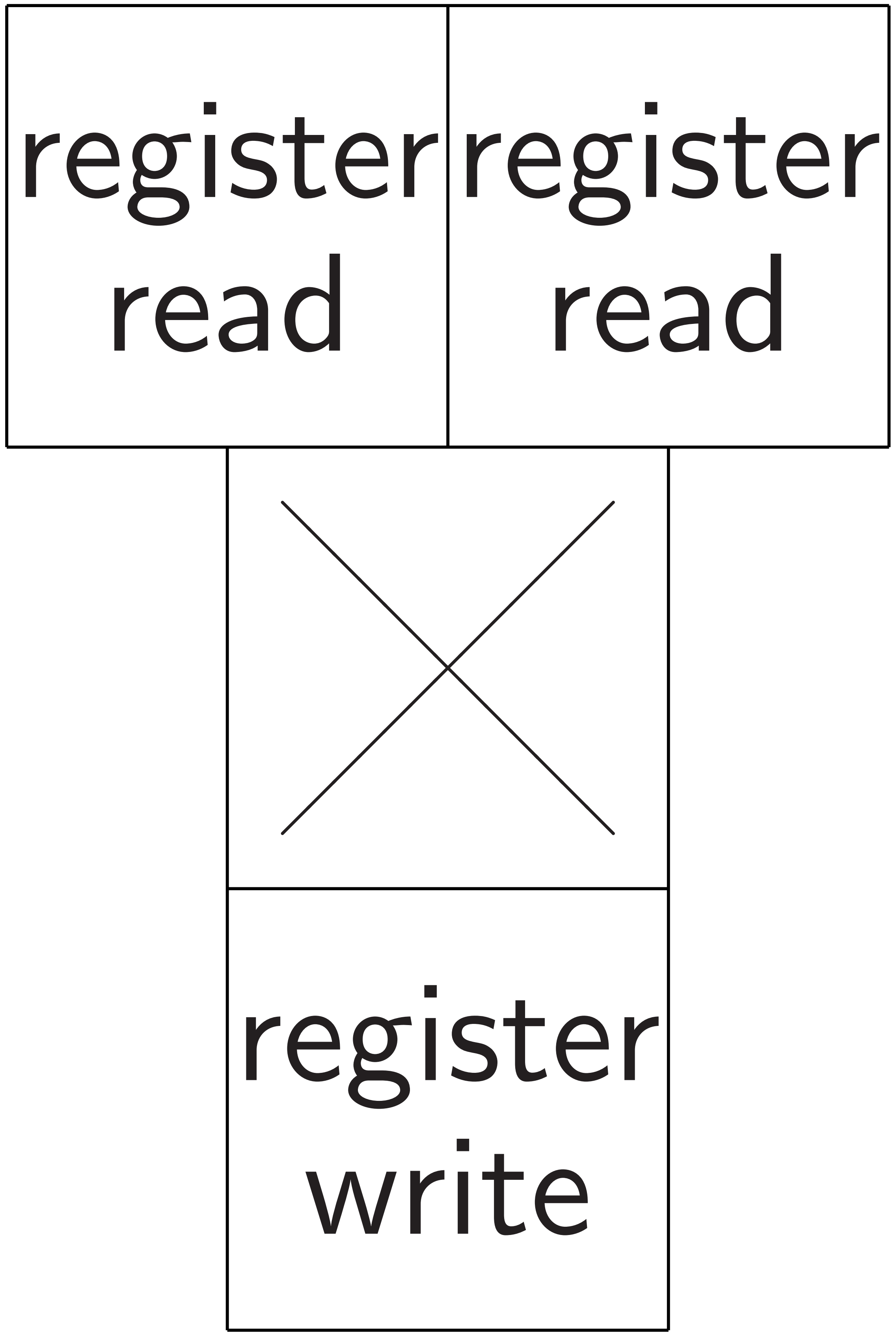




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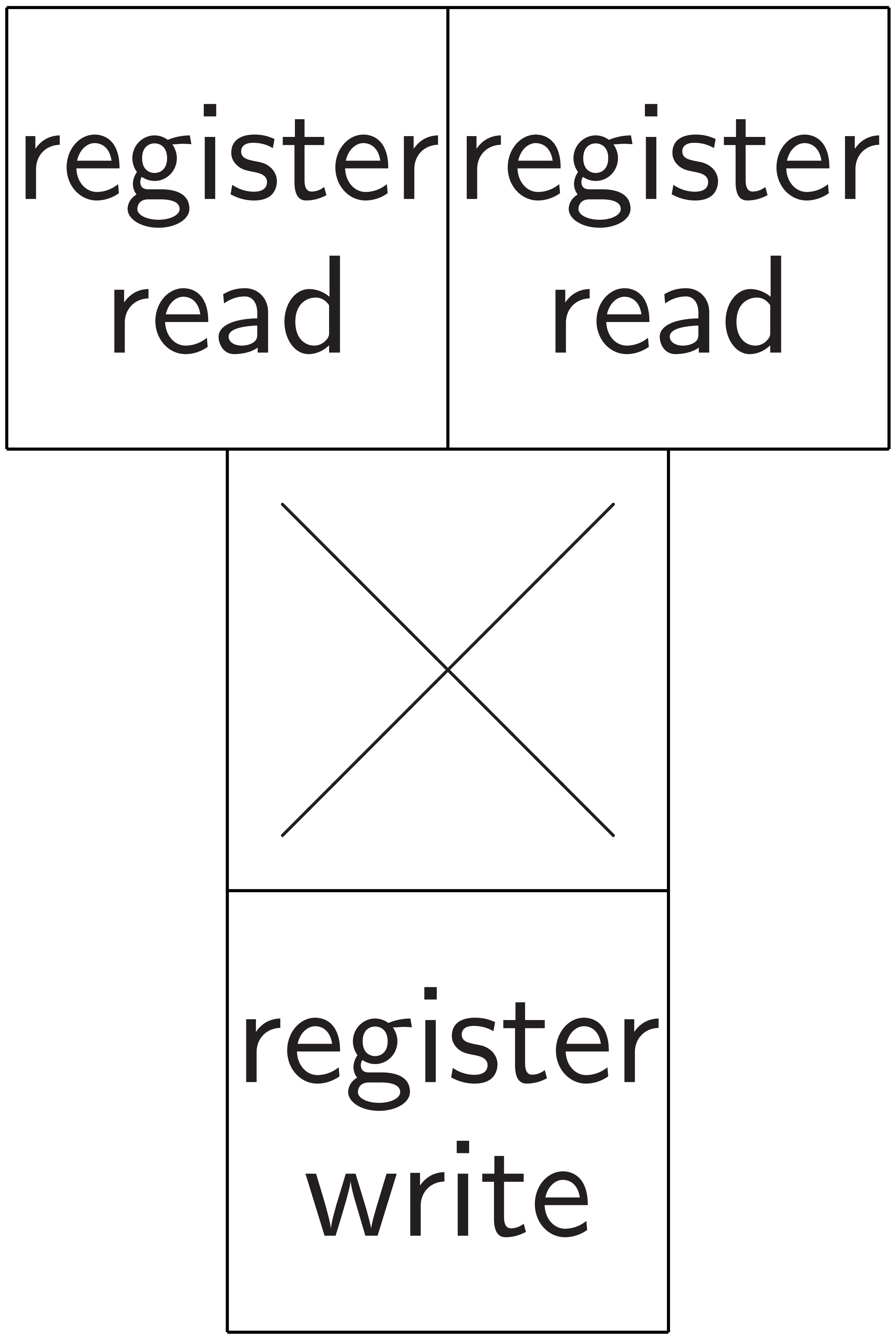




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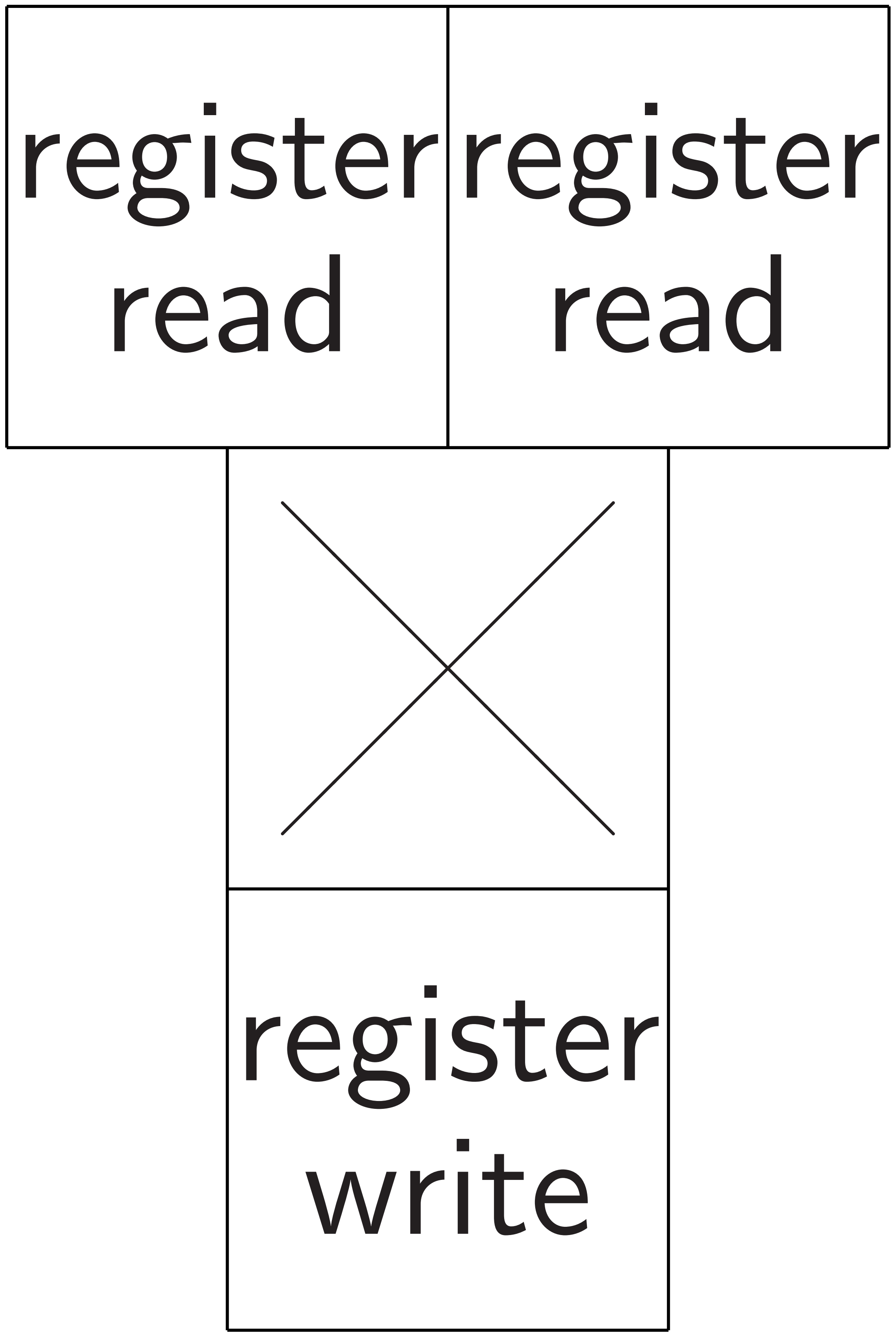




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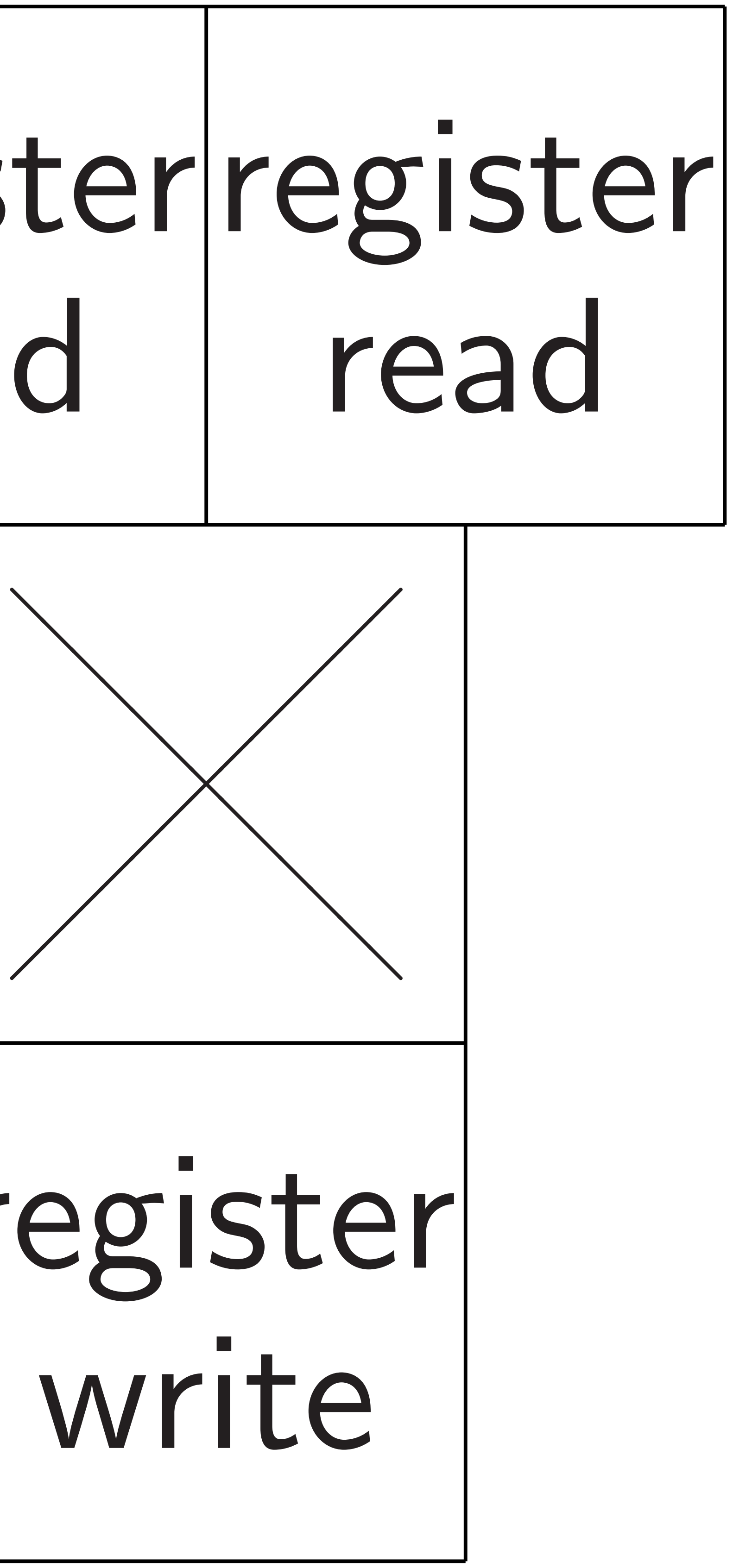




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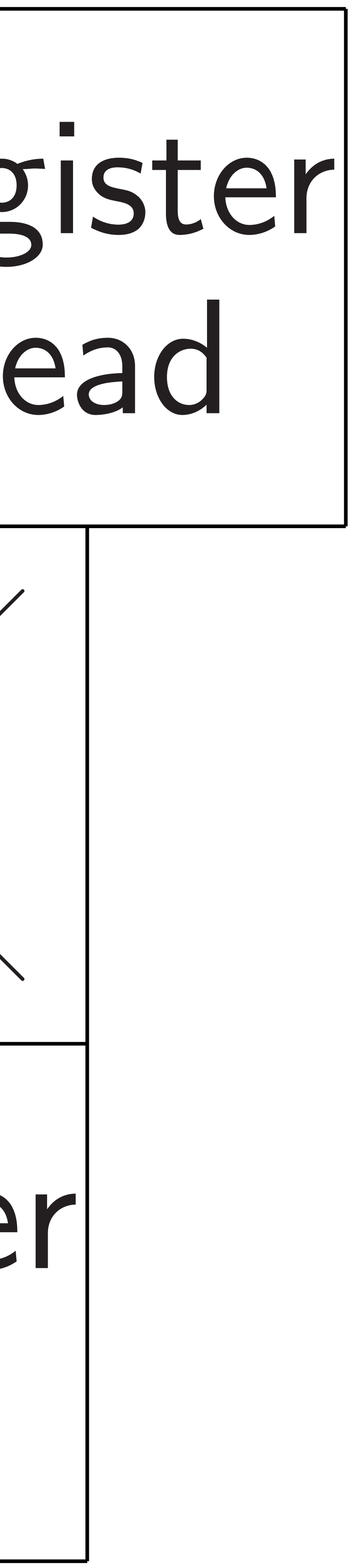




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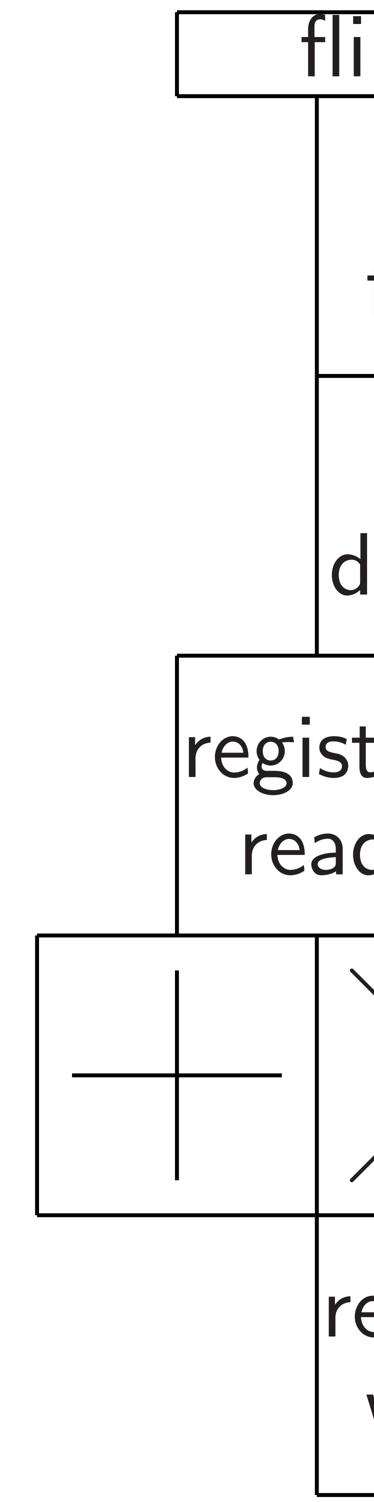


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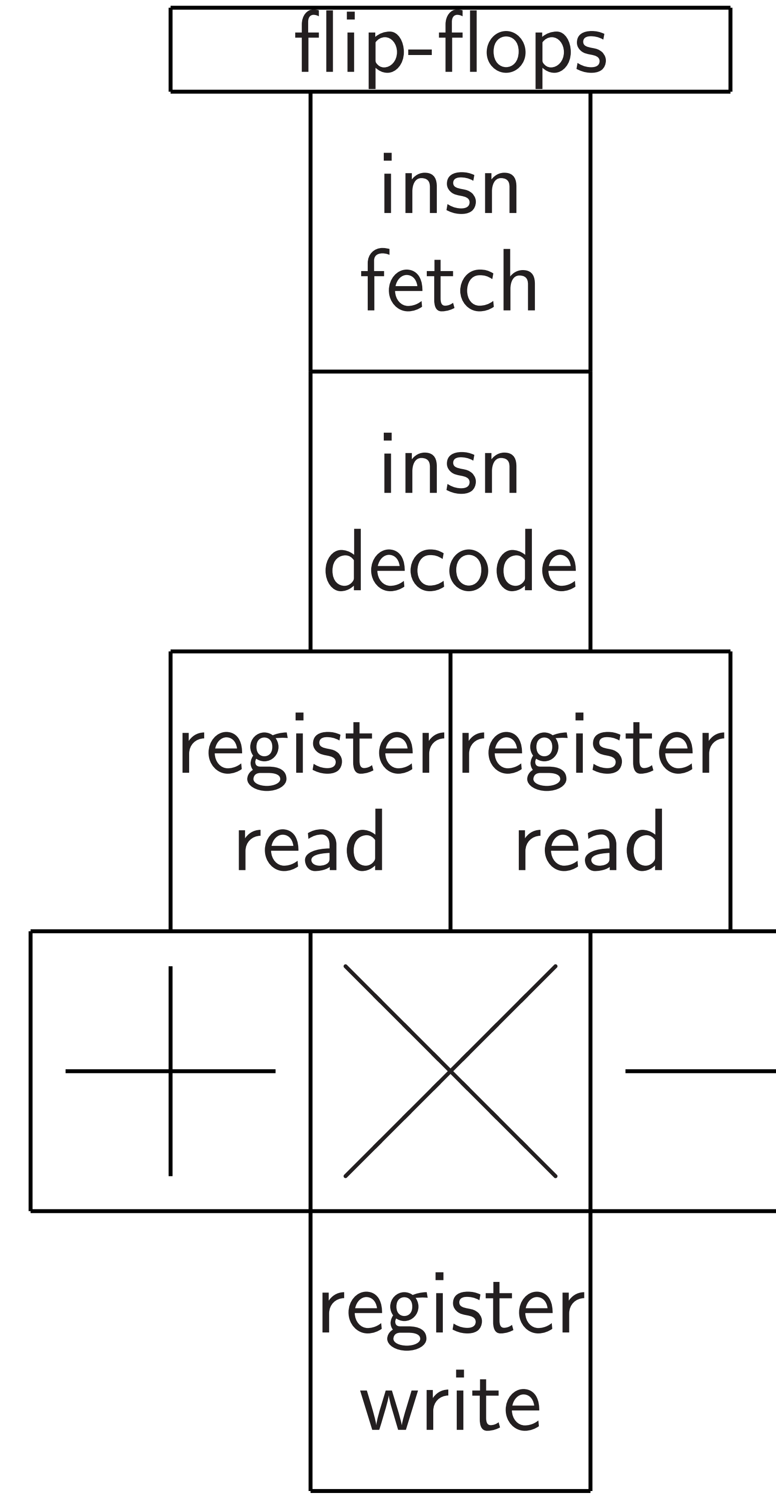


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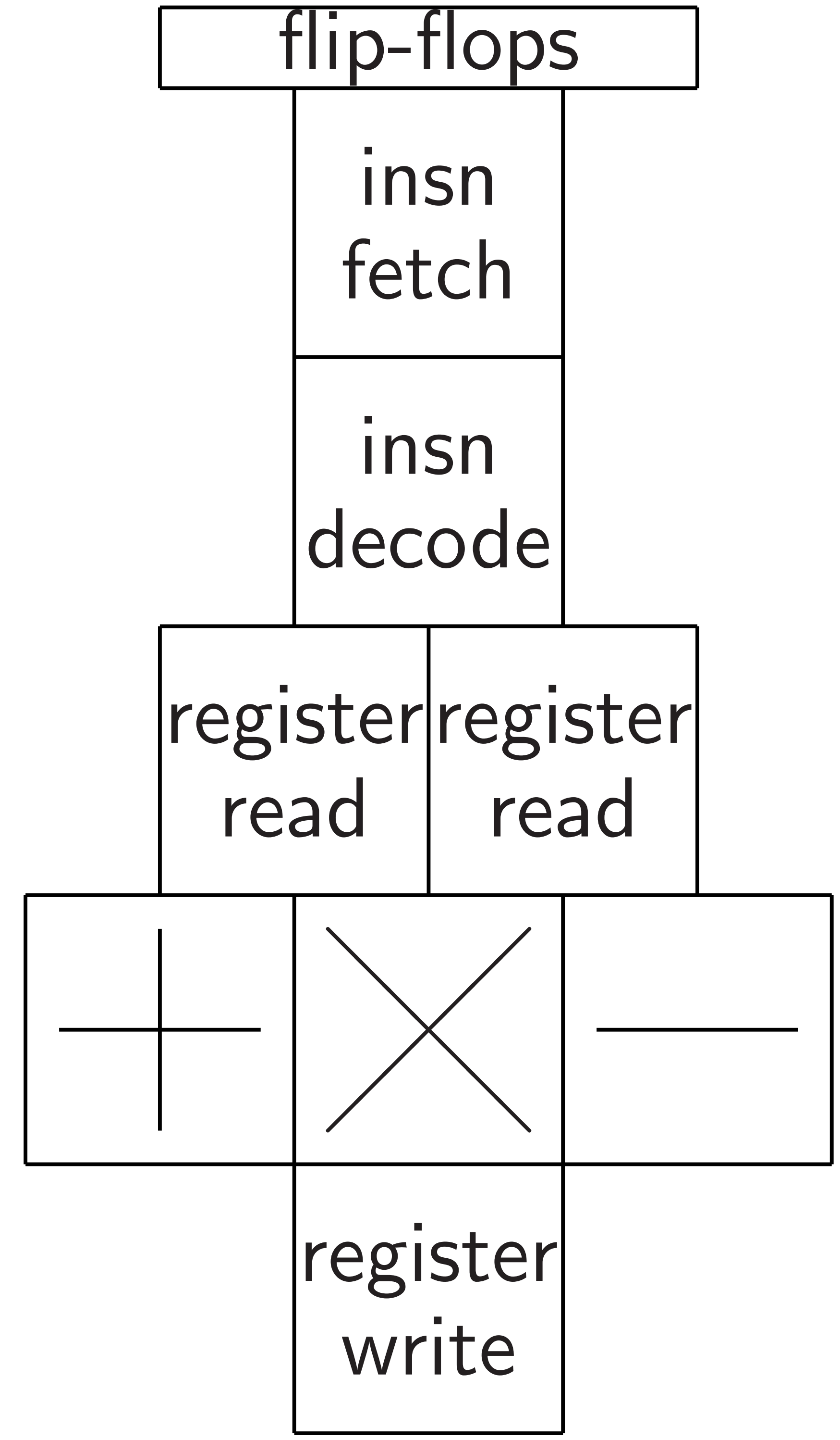


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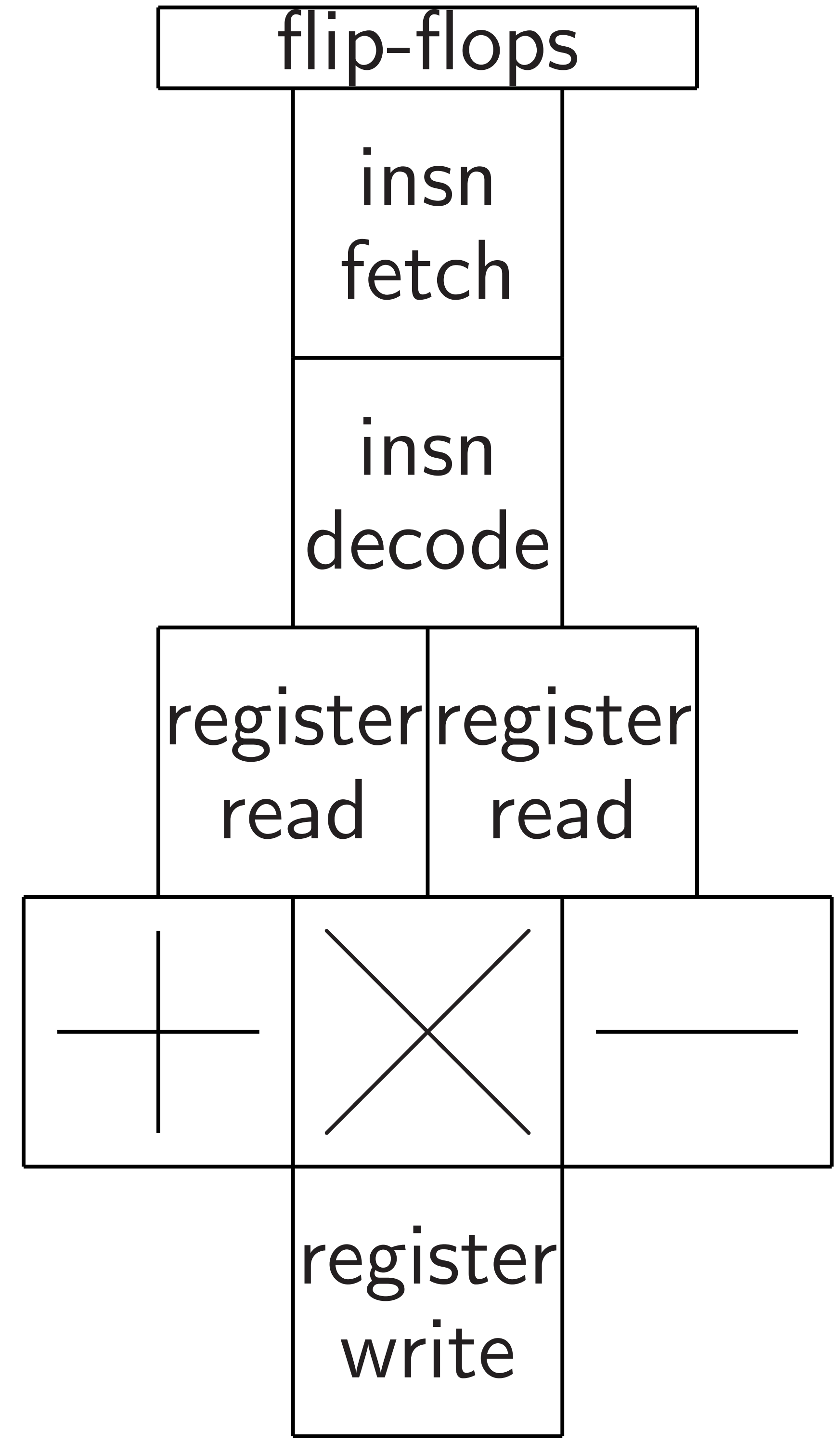


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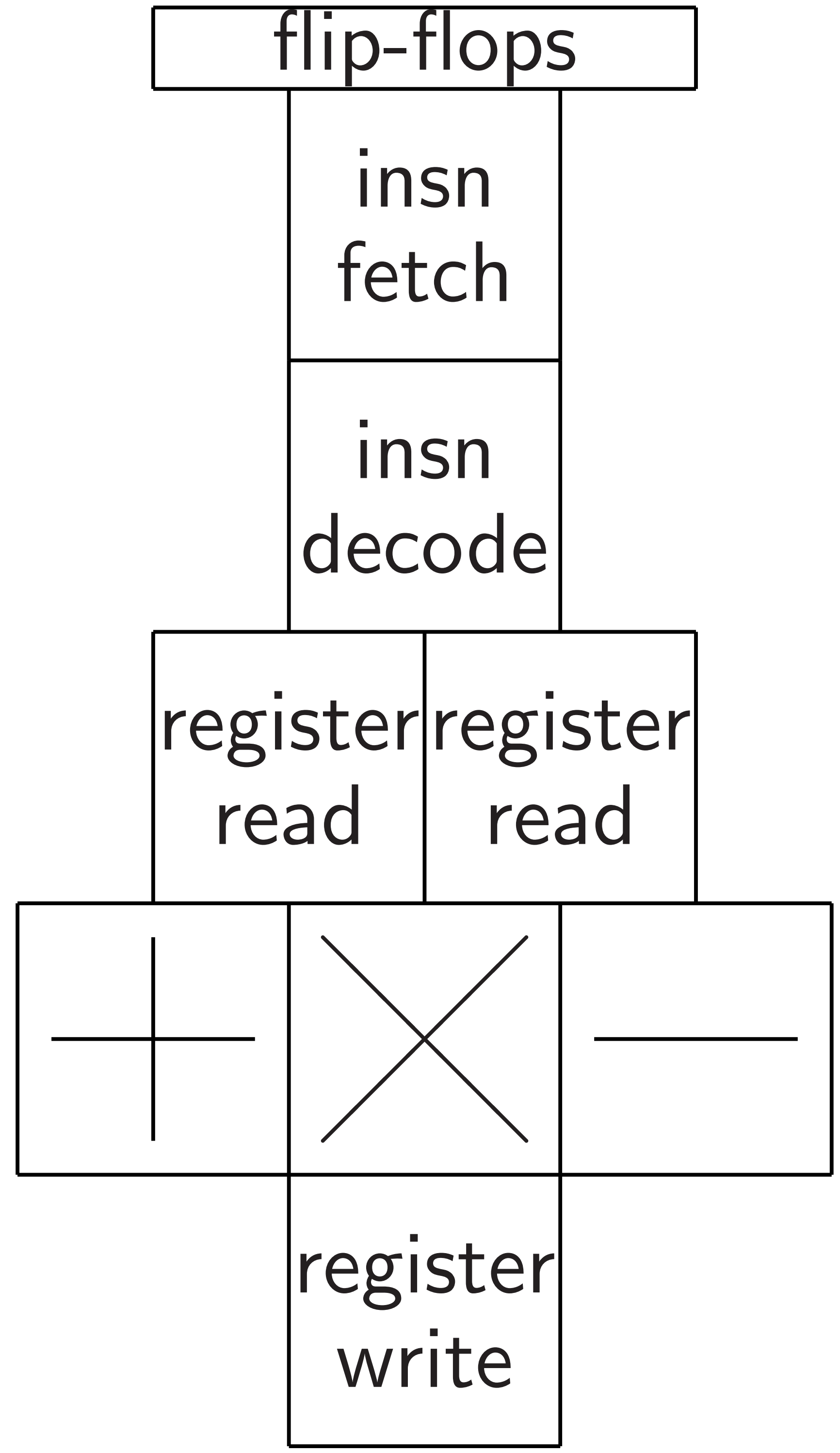


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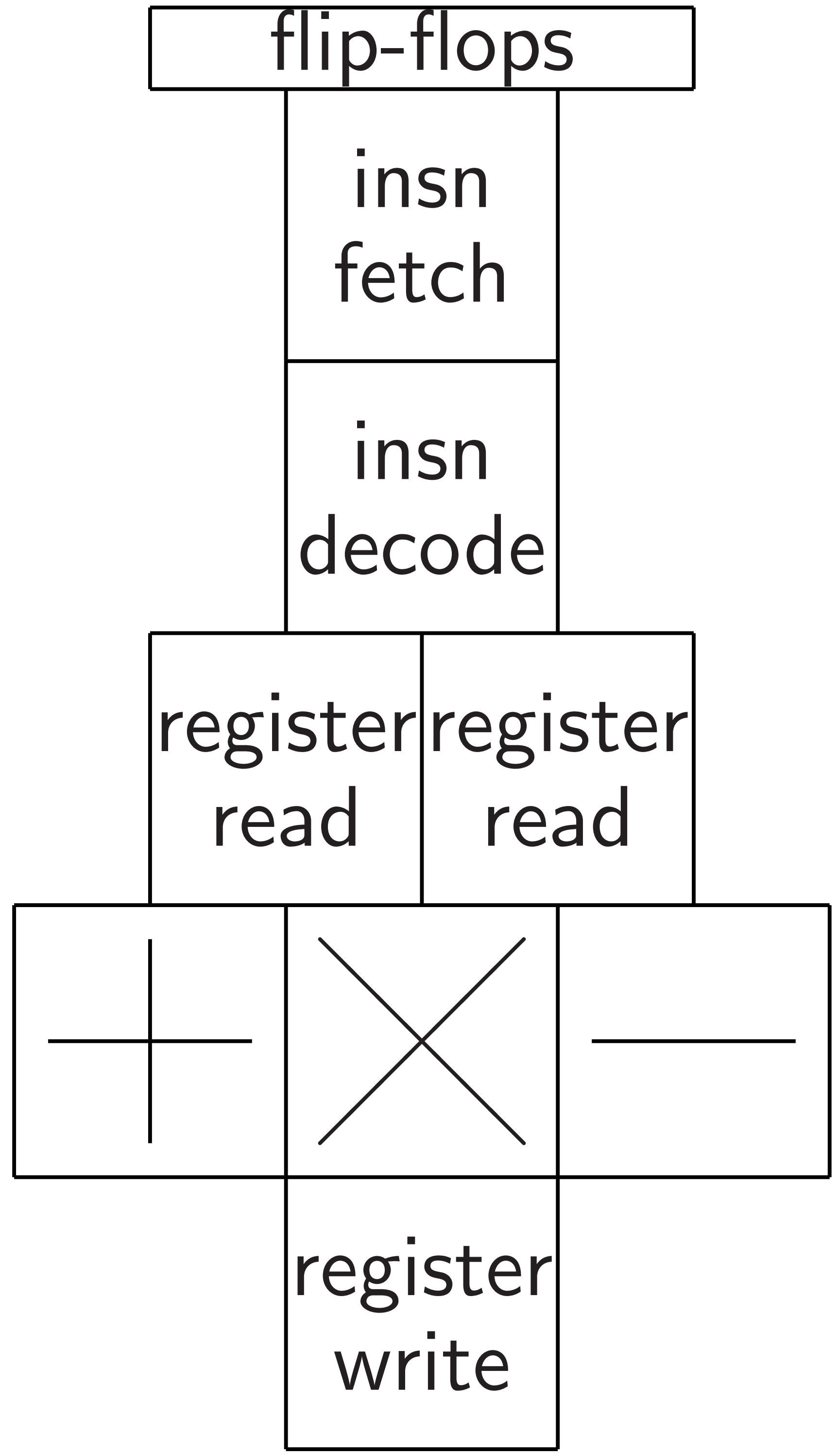


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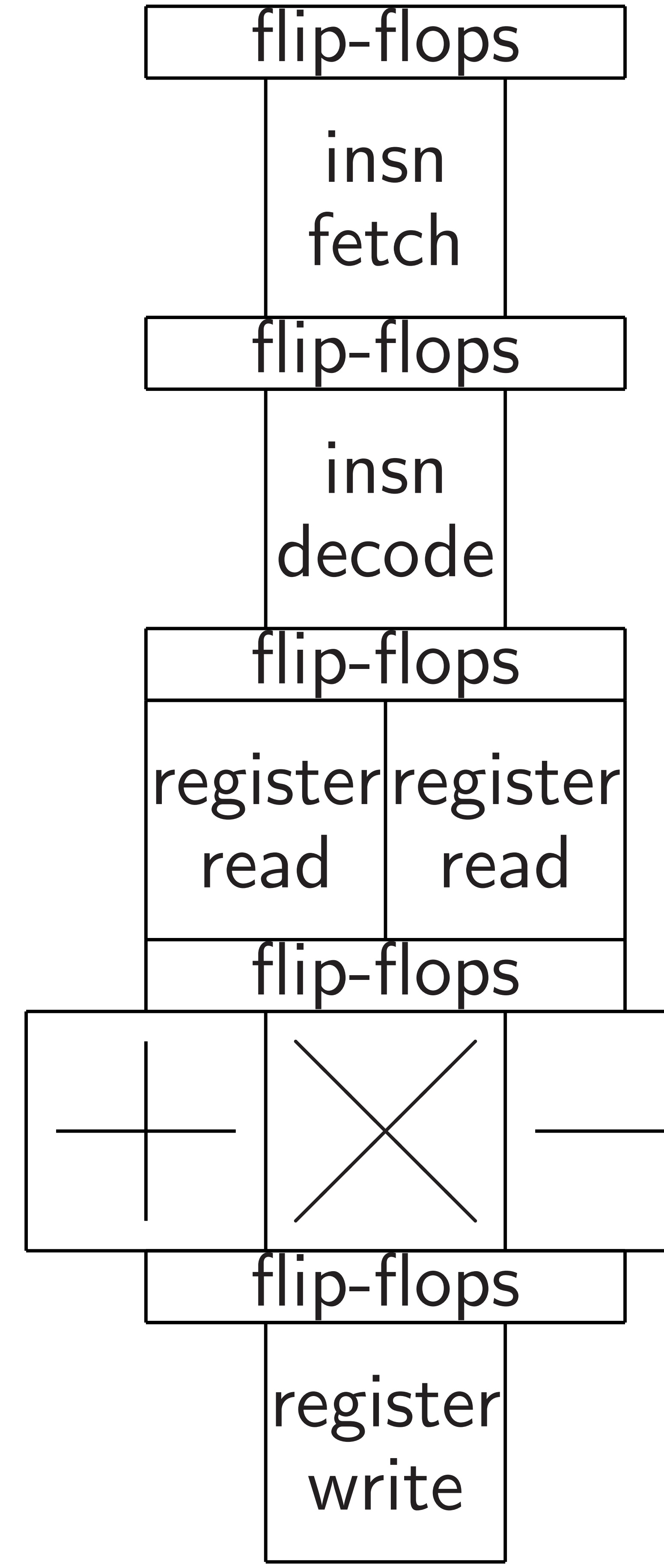

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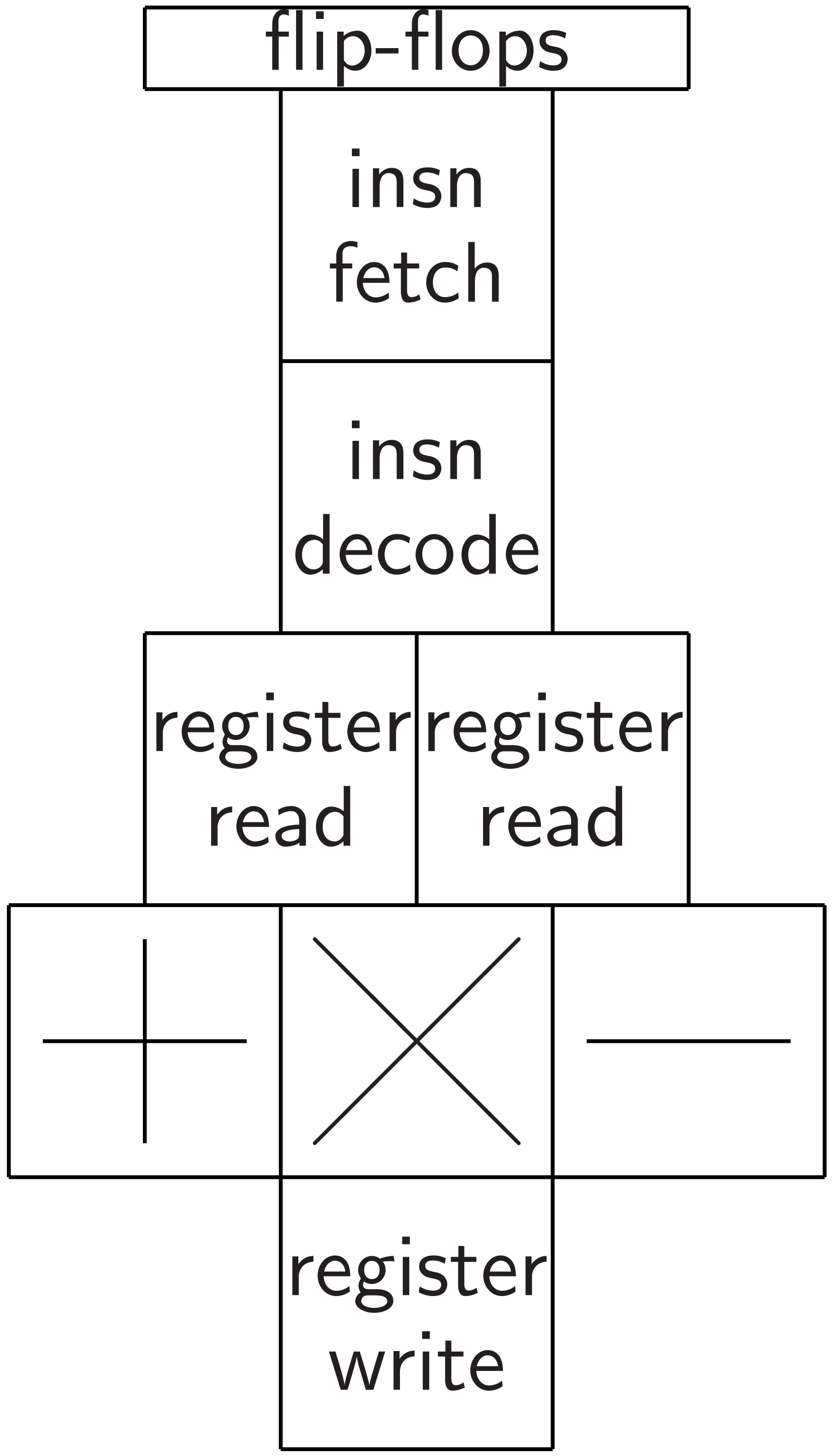


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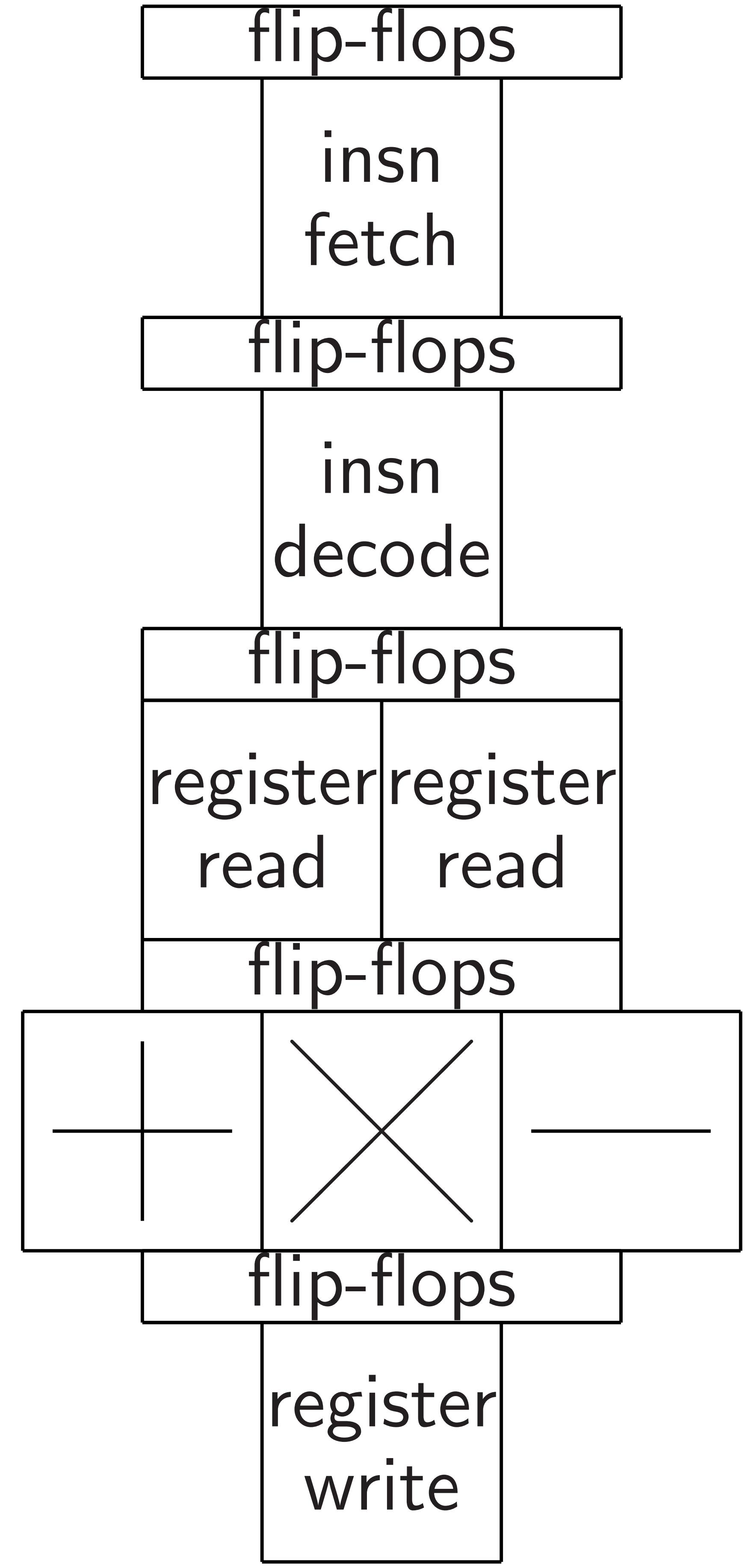


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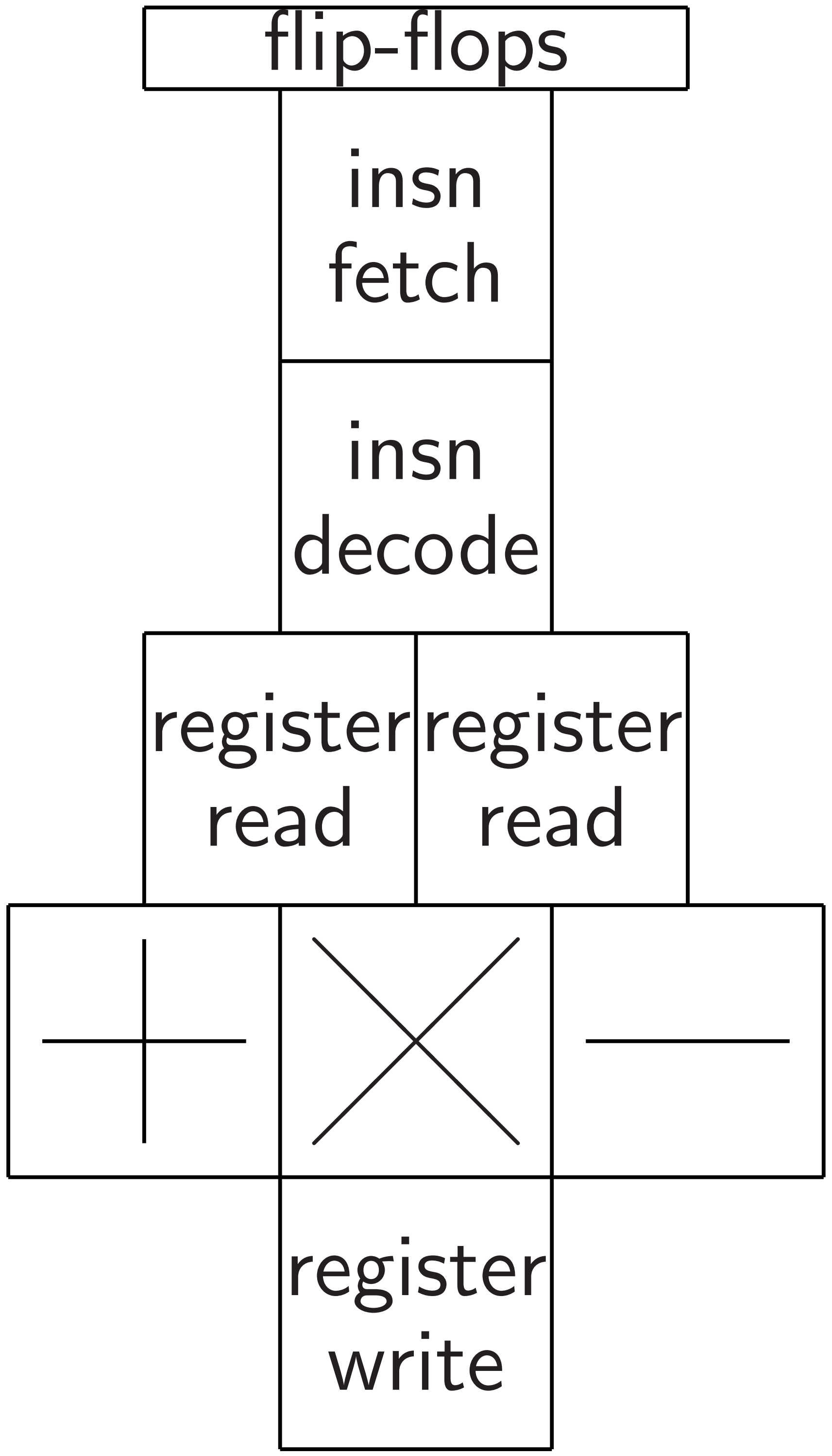


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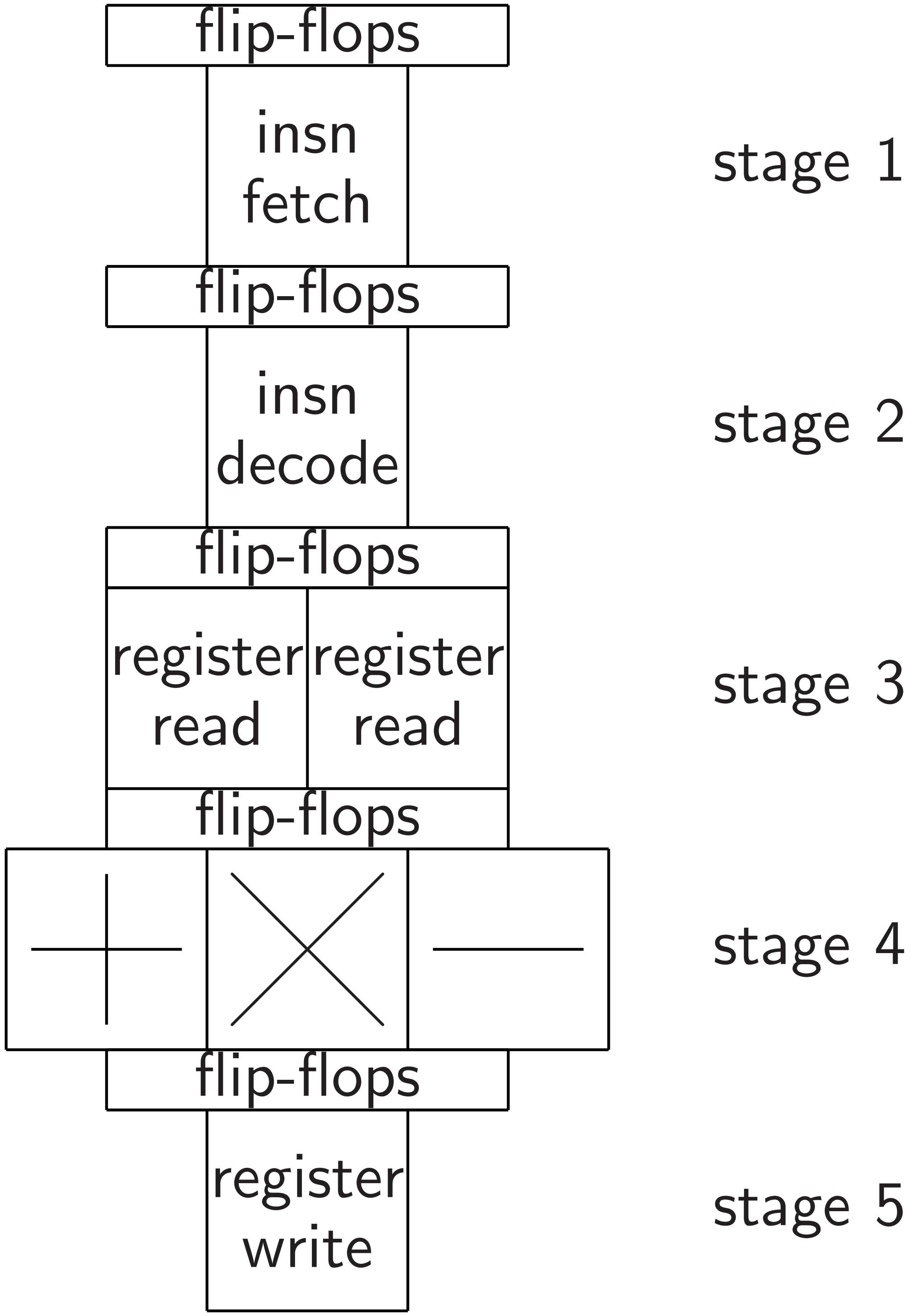

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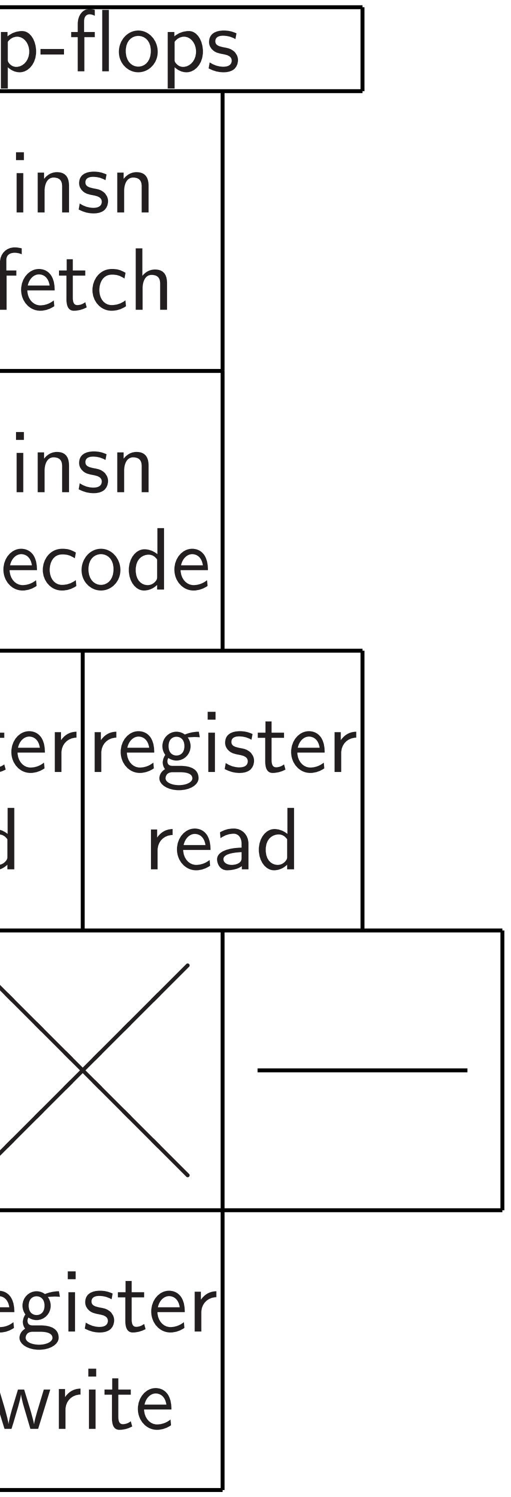


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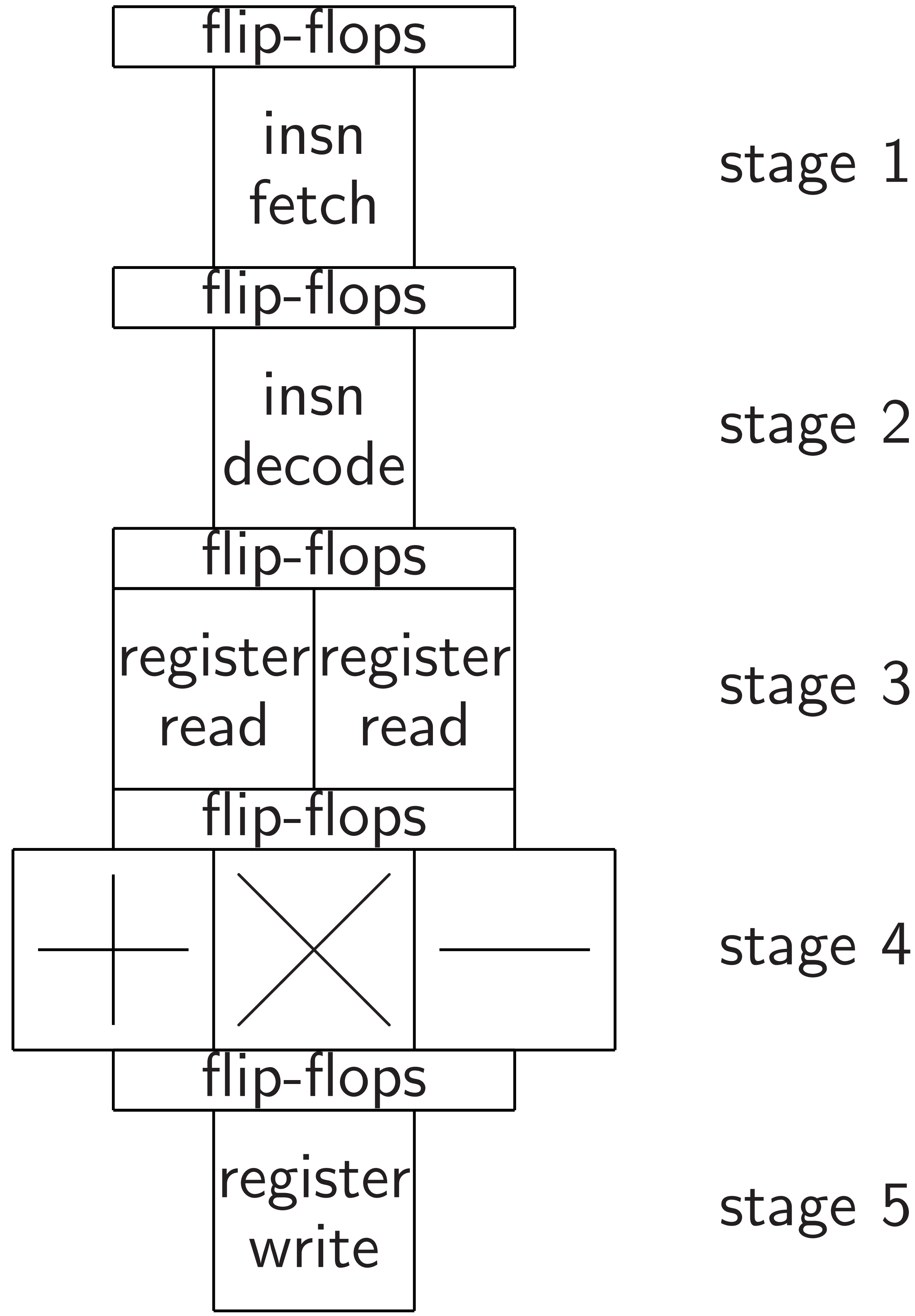


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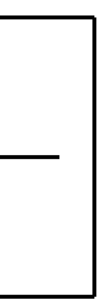

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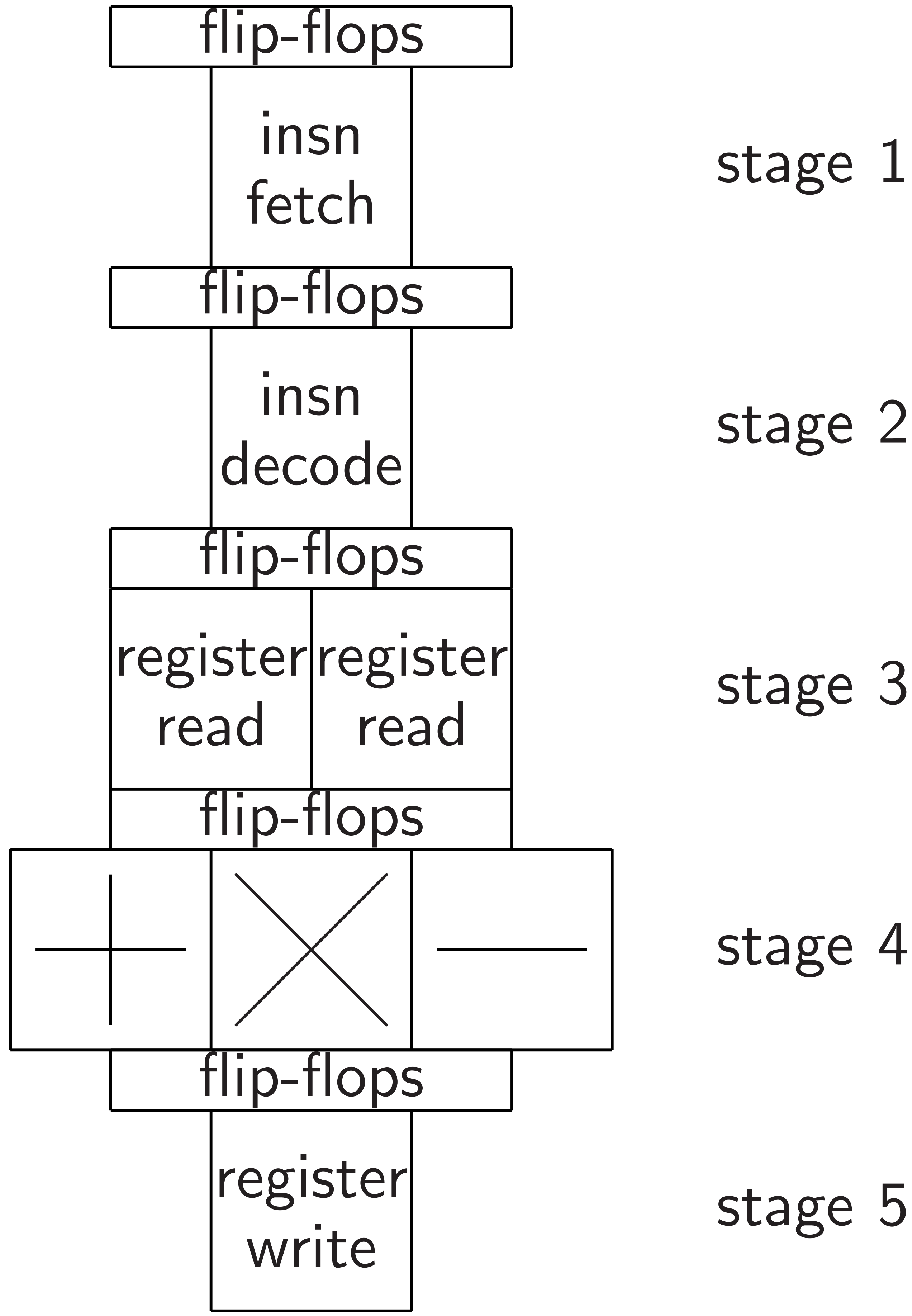


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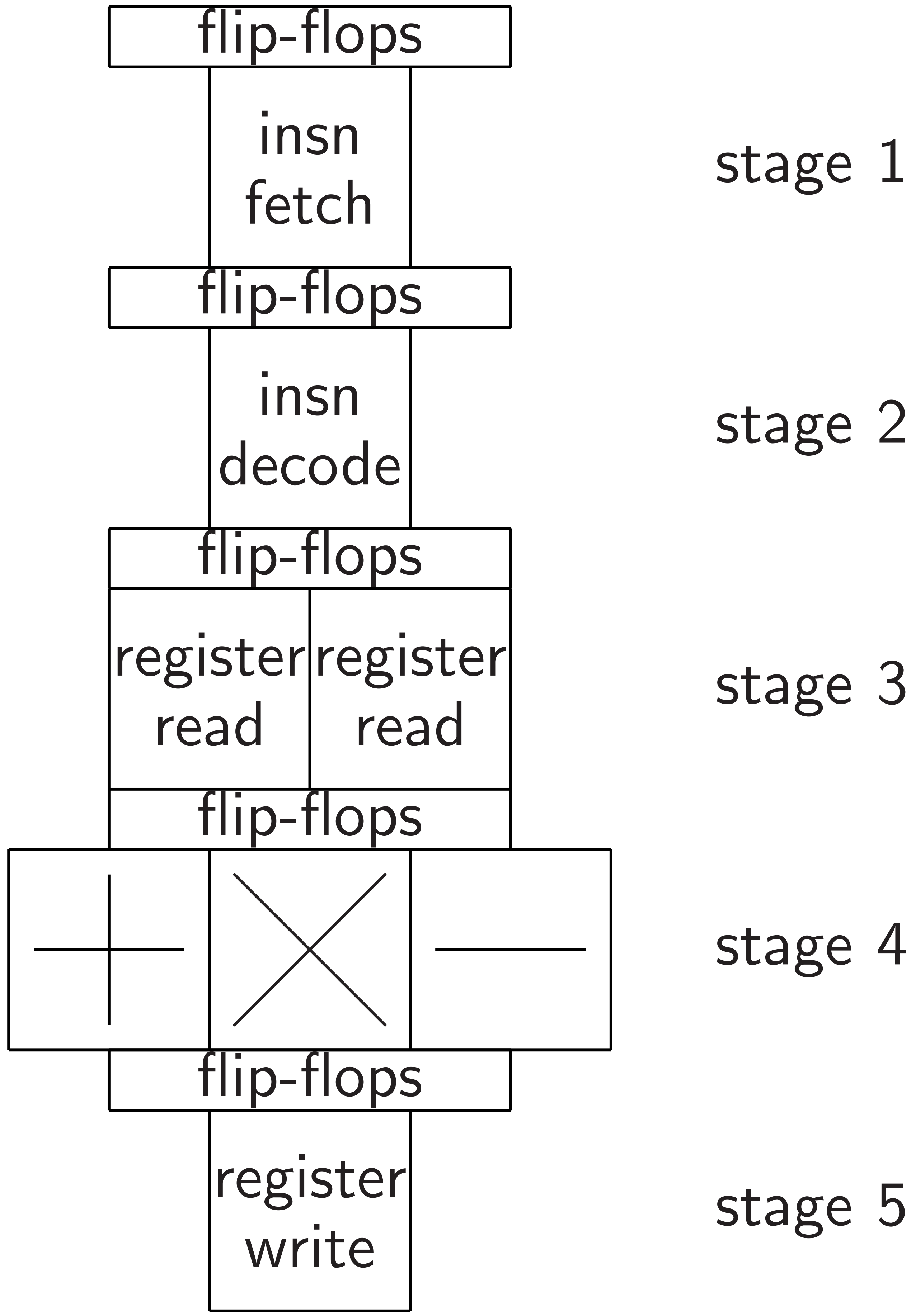




Goal: Stage $n$ handles instru
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Instruction fetch
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e.g., stall on read-after-write

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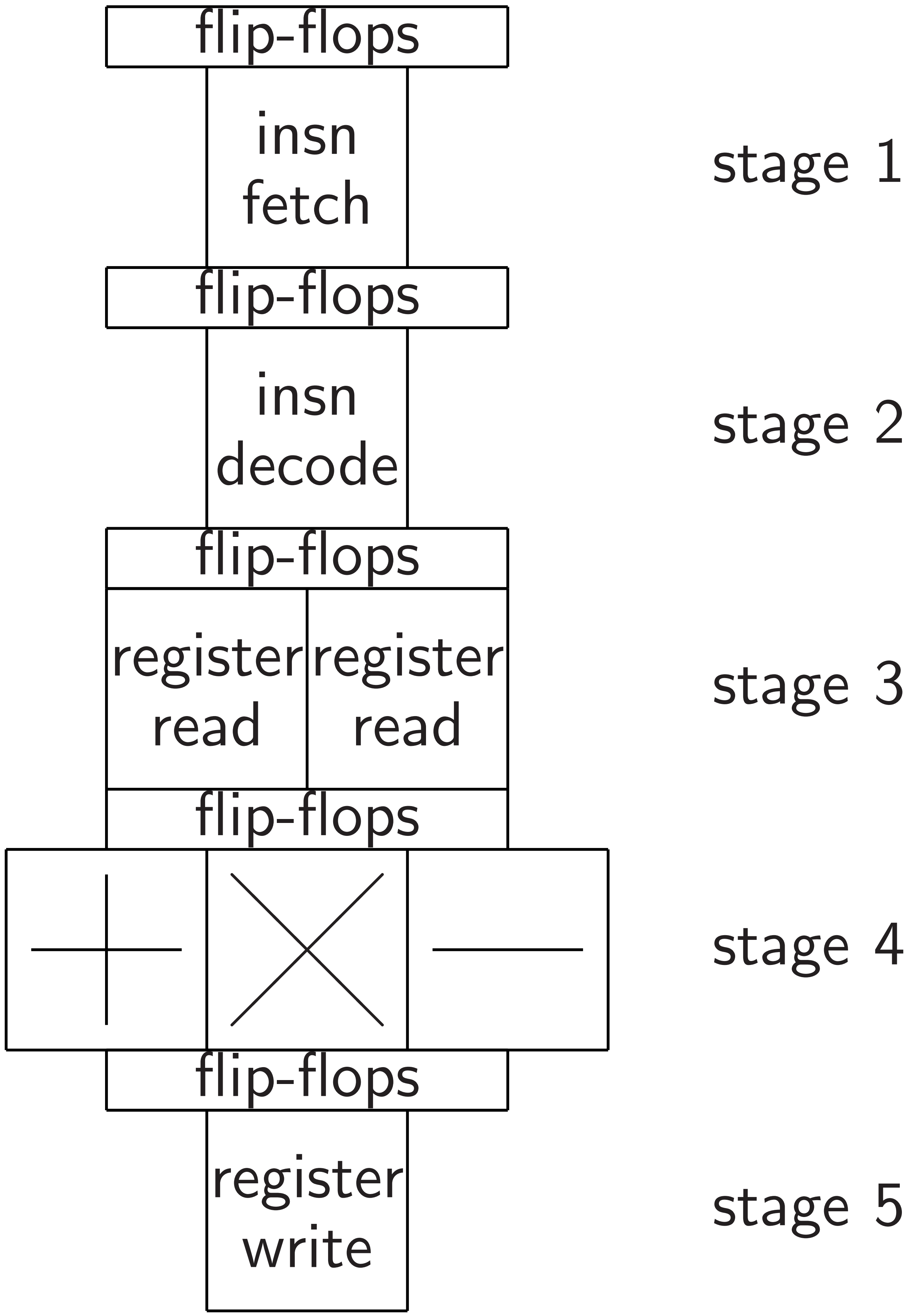




Goal: Stage $n$ handles instruction one tick after stage $n - 1$.

Instruction fetch reads next instruction, feeds $p'$ back, sends instruction.

After next clock tick, instruction decode uncompresses this instruction, while instruction fetch reads another instruction.

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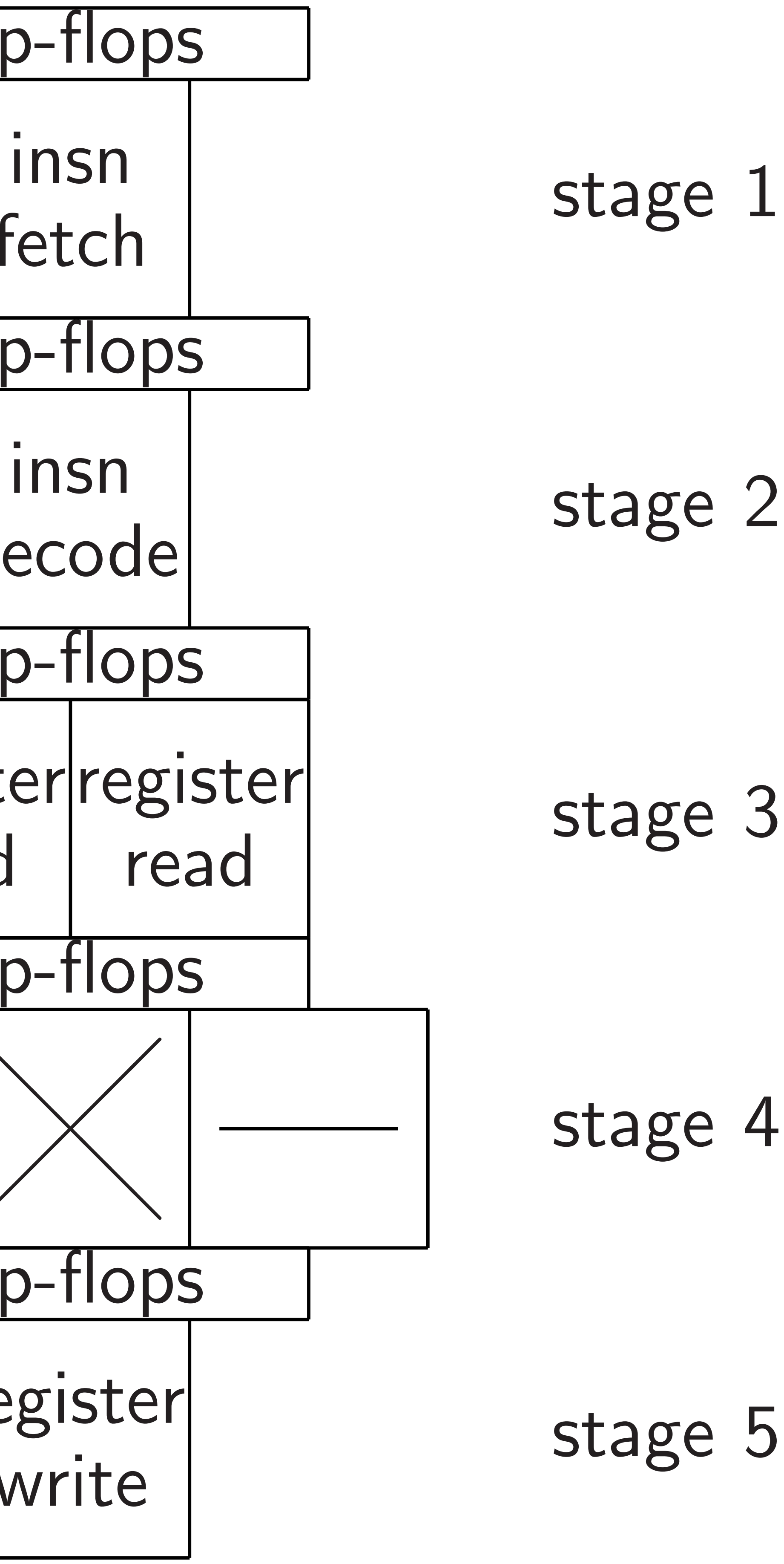




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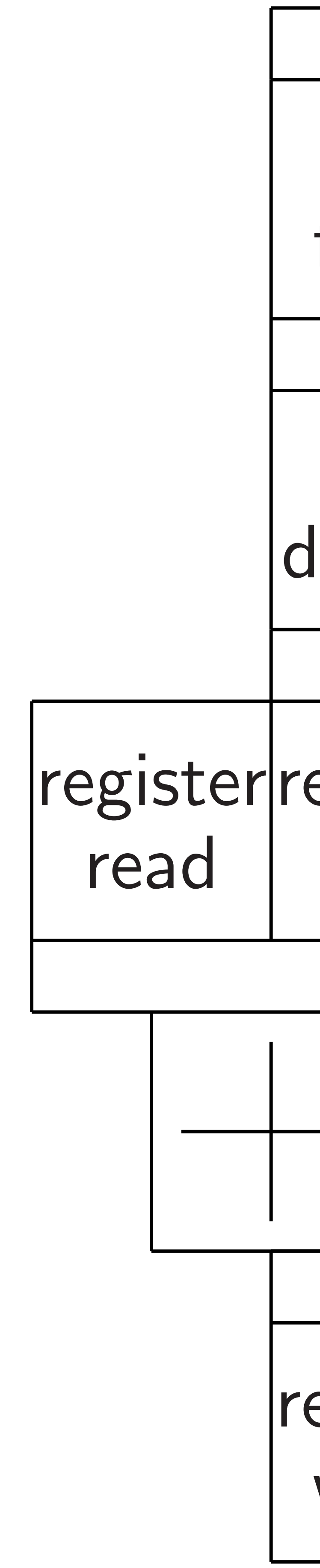

s faster clock:

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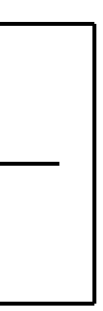
stage 3



stage 4

stage 5



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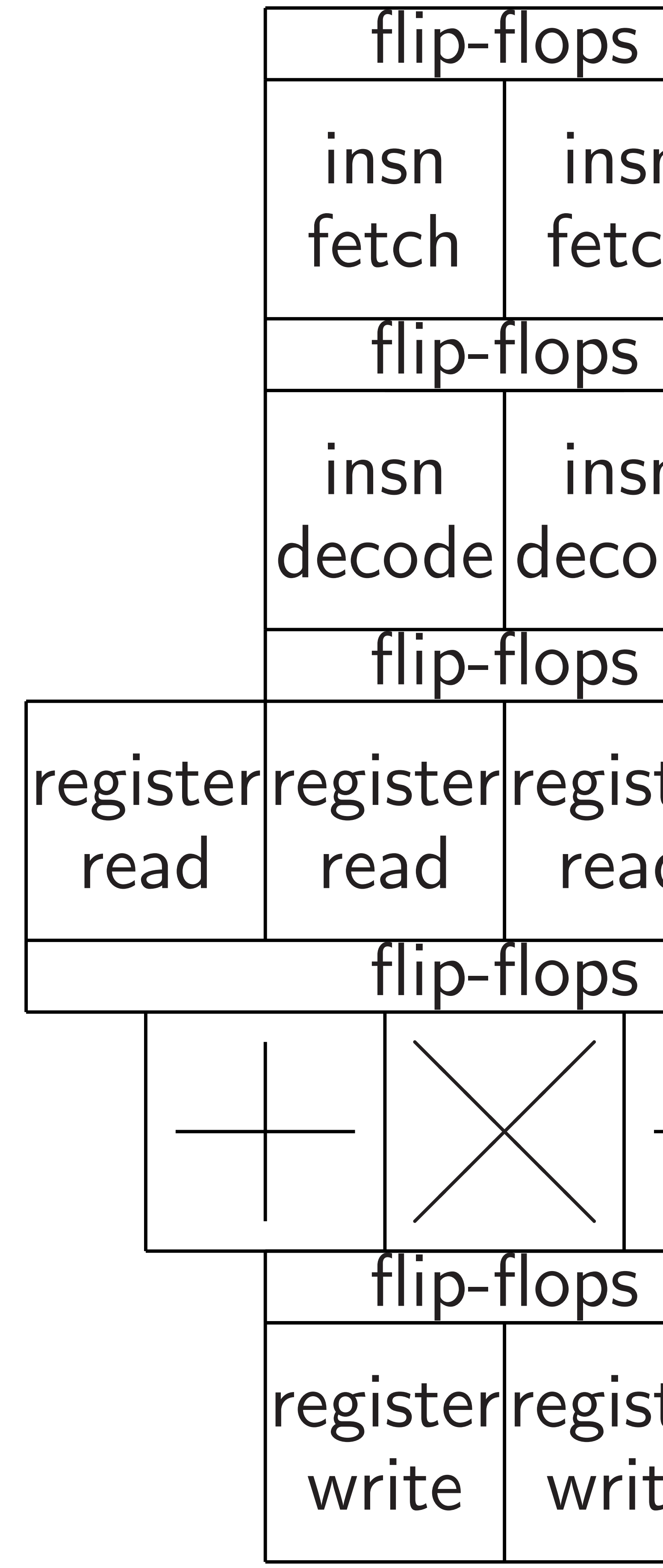

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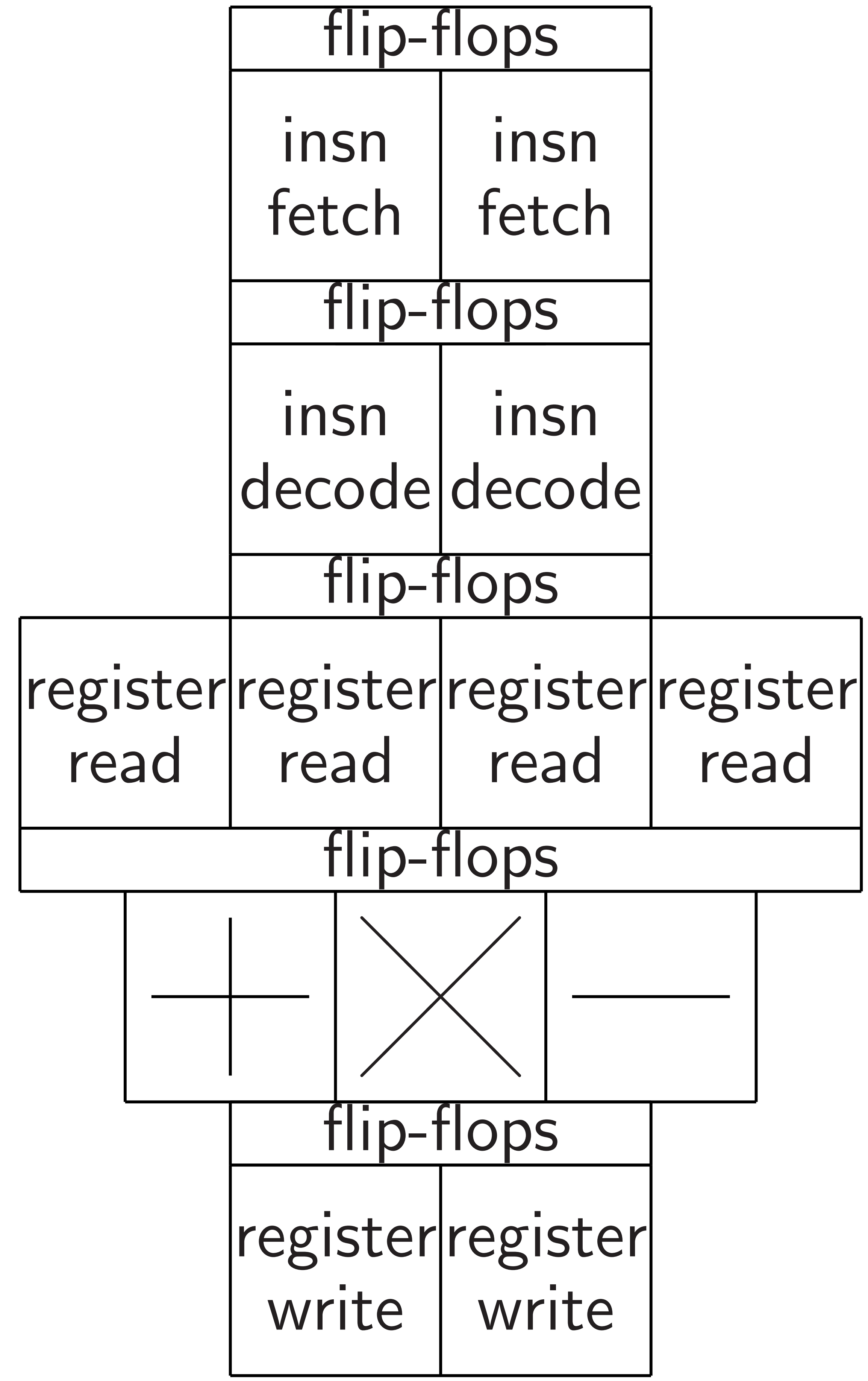

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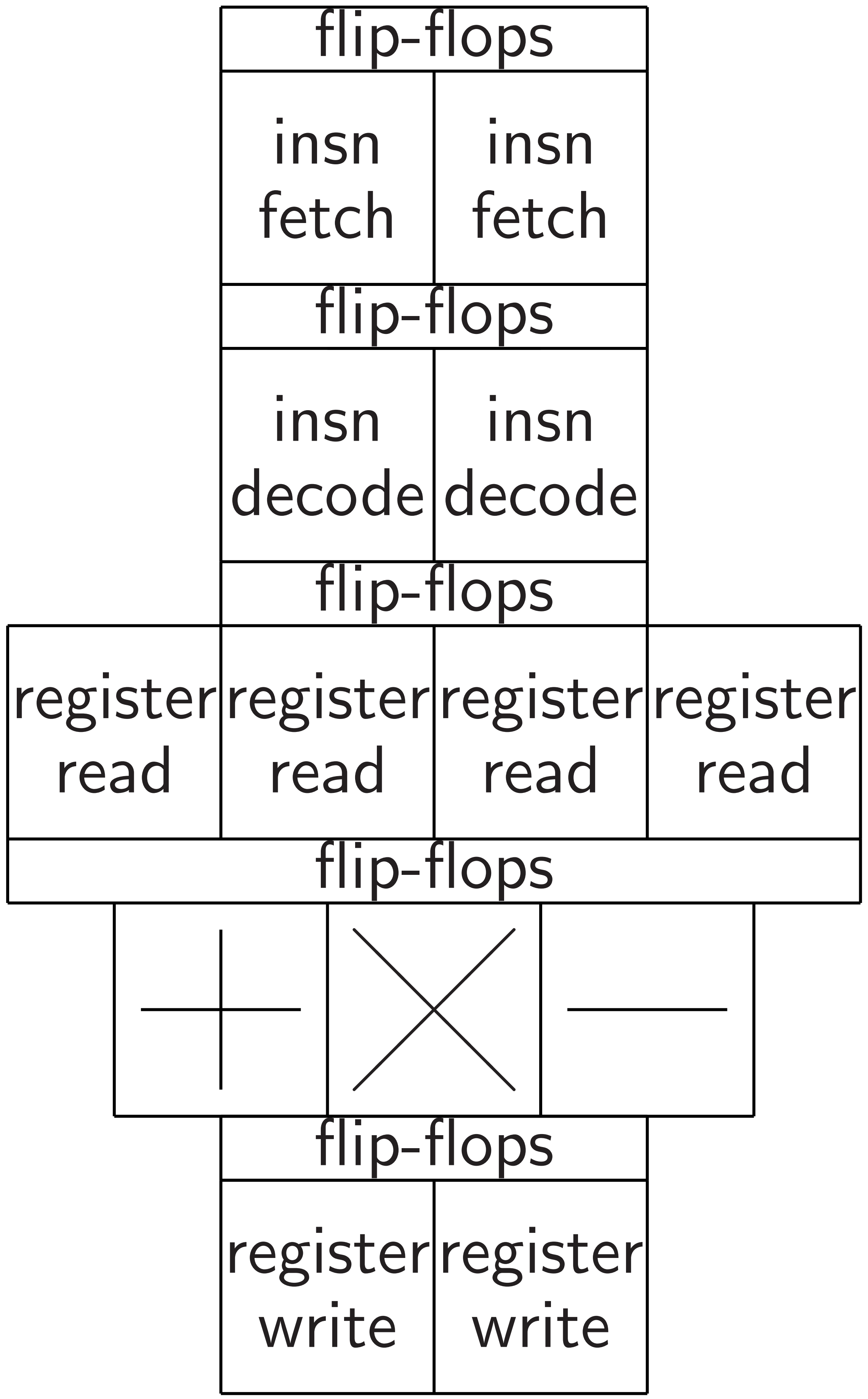

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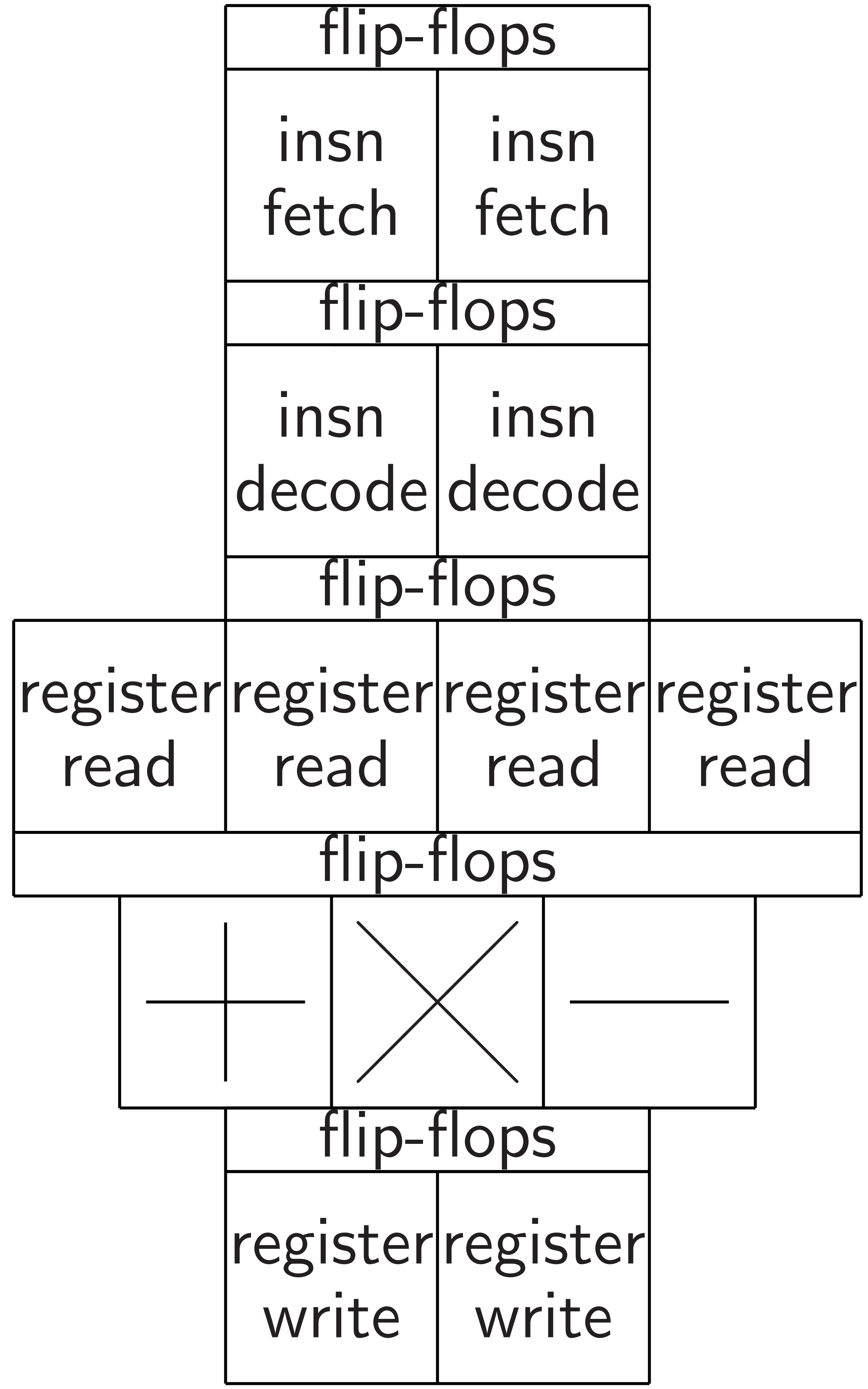


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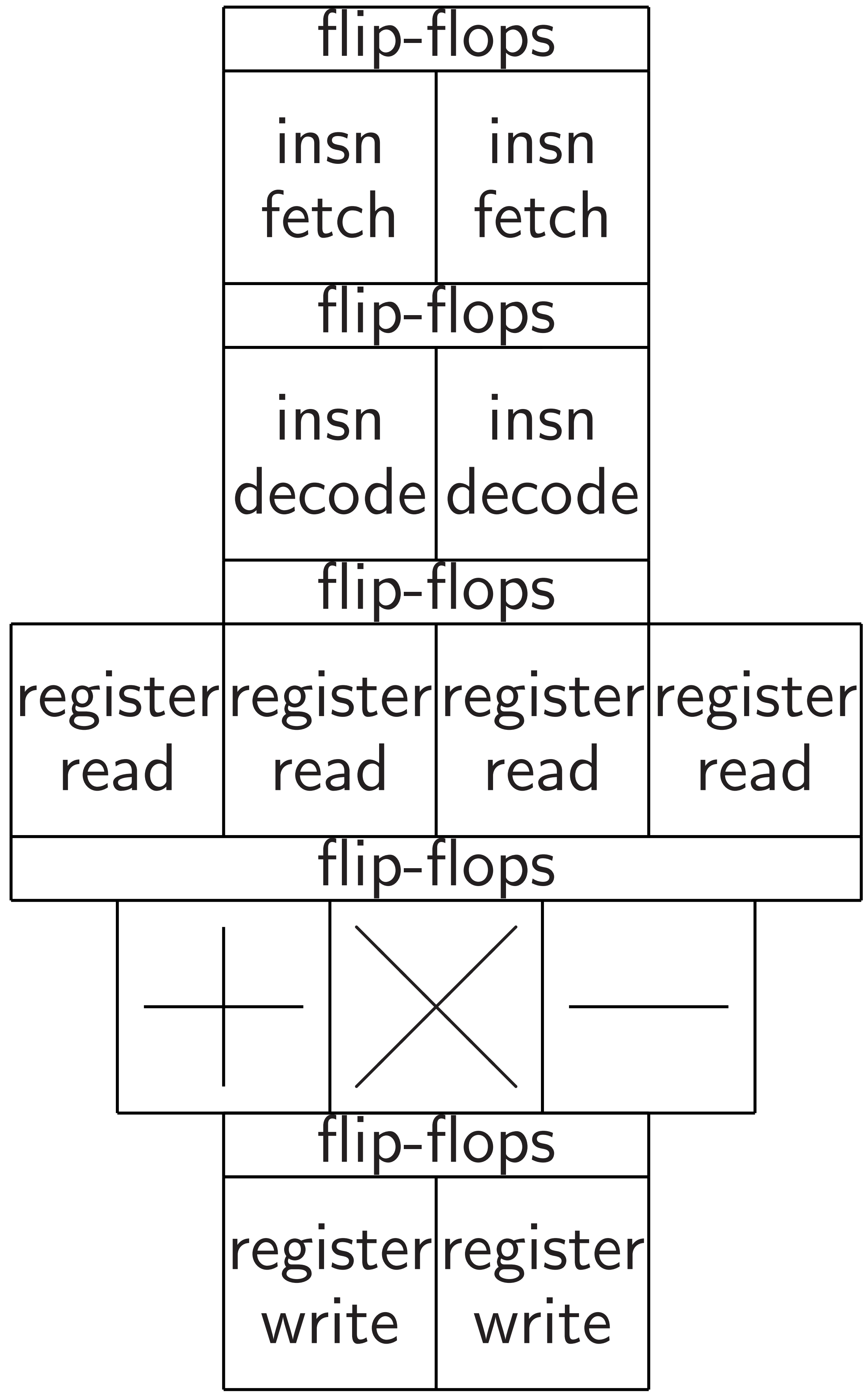


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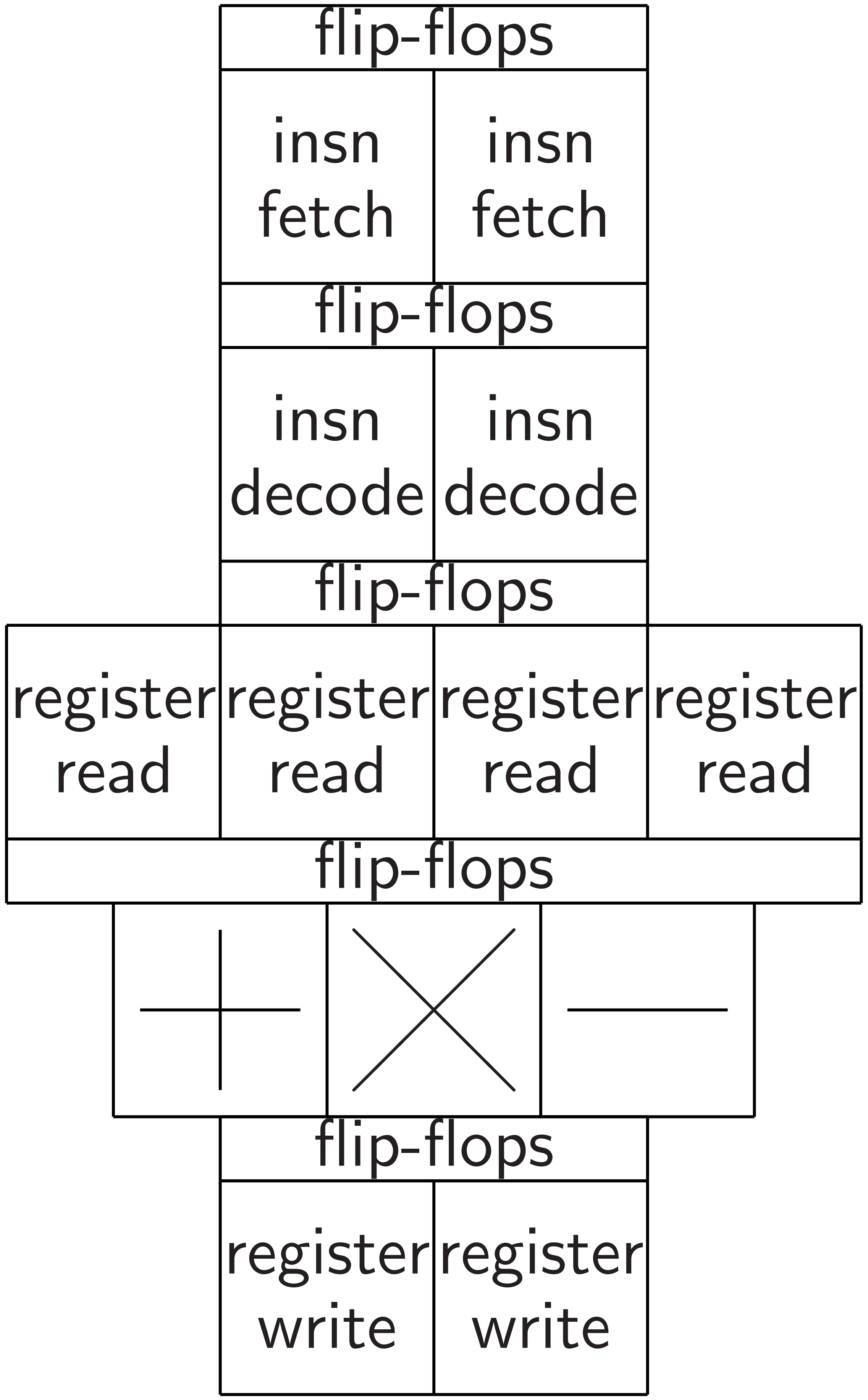




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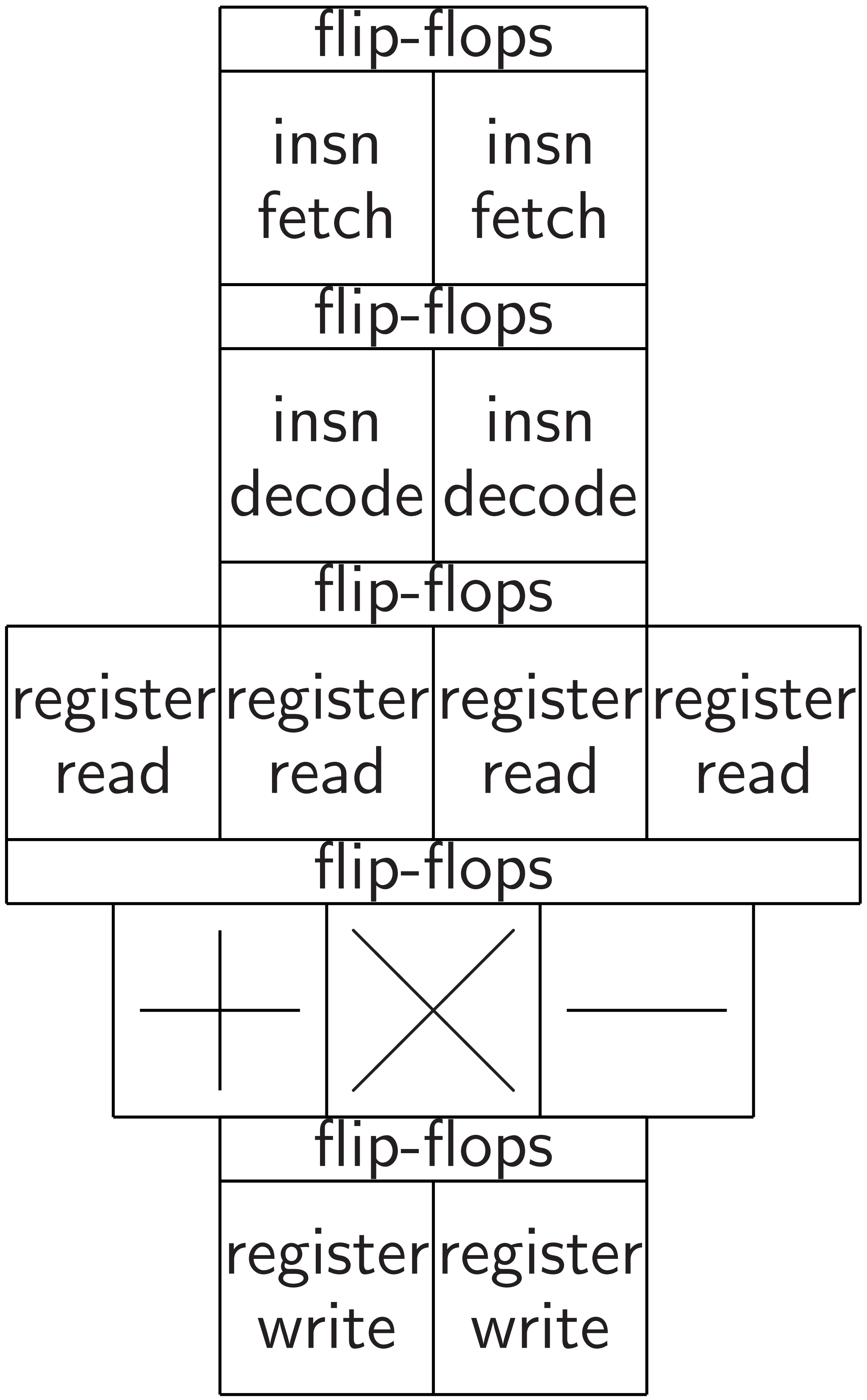




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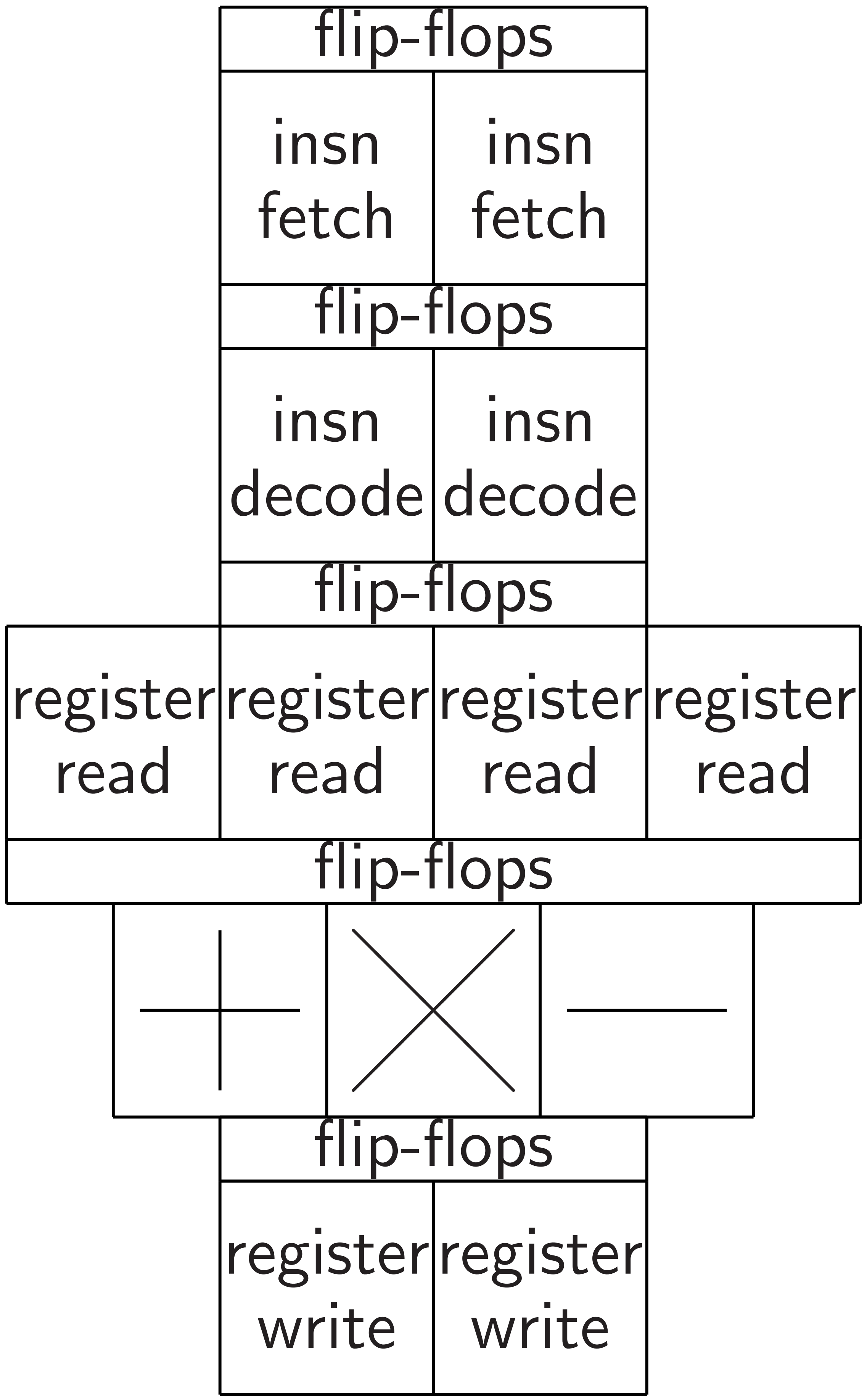




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Benefit: Amortizes insn circuits.

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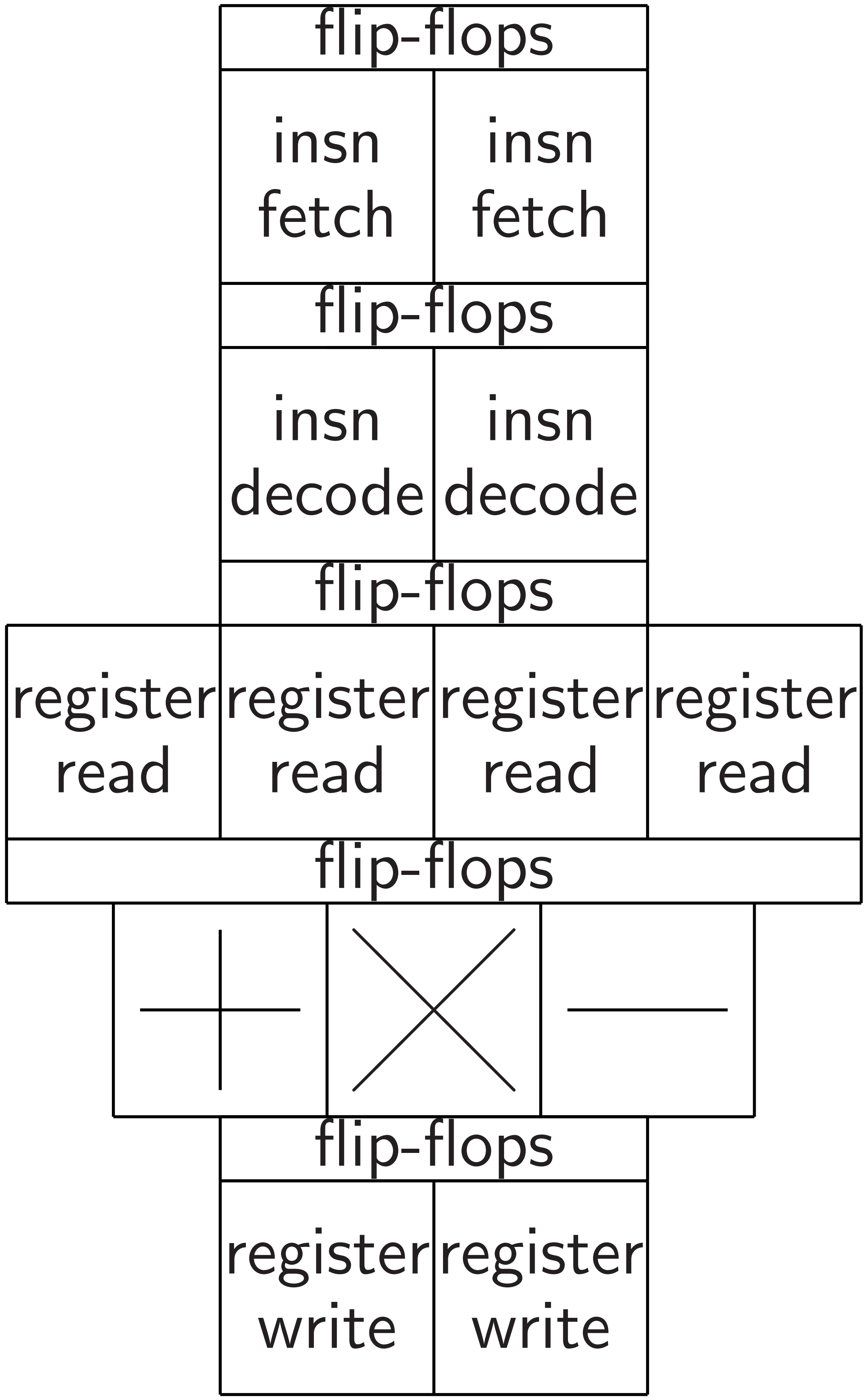




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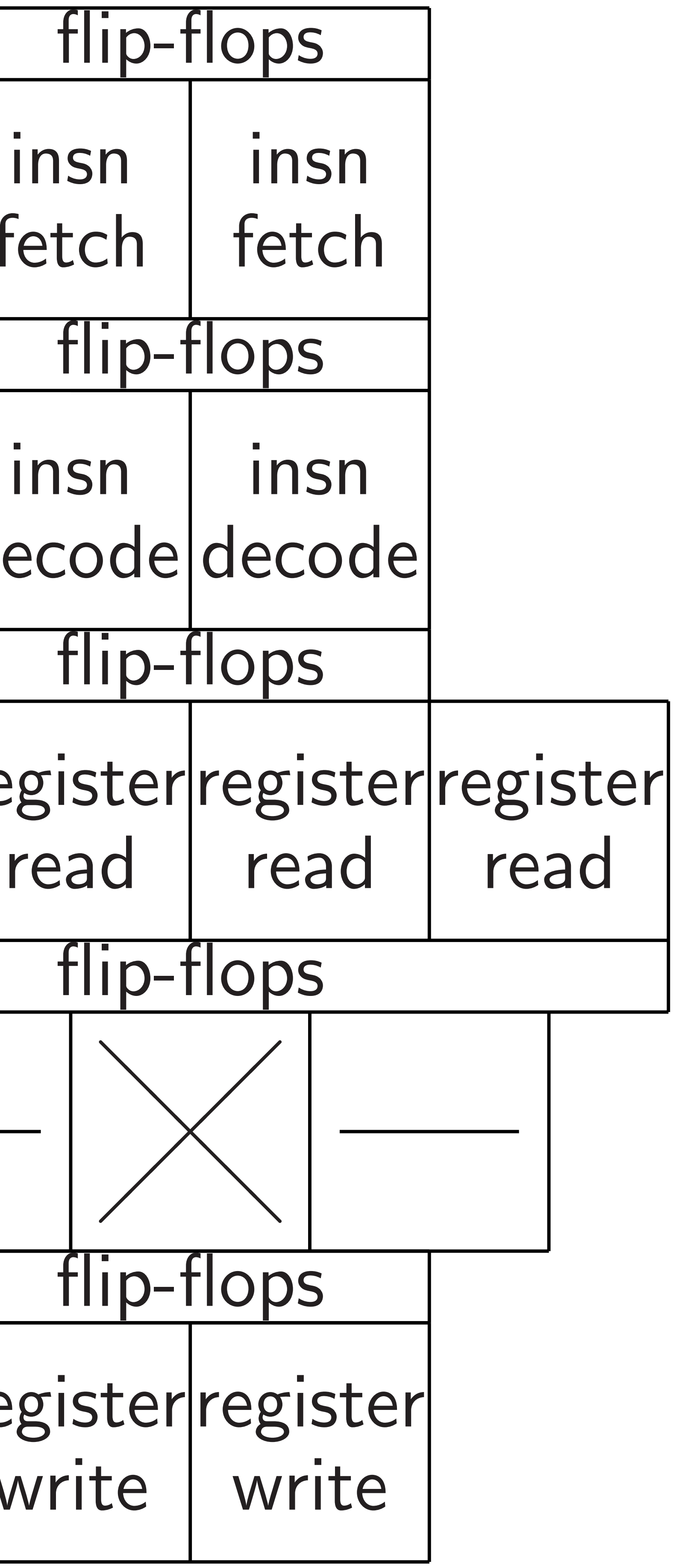




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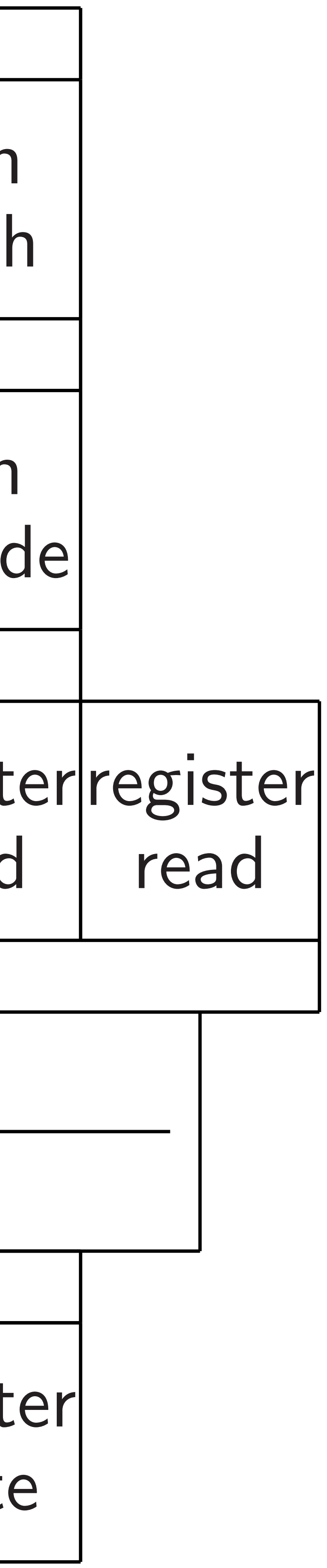




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## Network on chip: the mesh

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Metric: seconds used by
circuit of area $n^{1+o(1)}$.

For simplicity assume $n = 4^k$.

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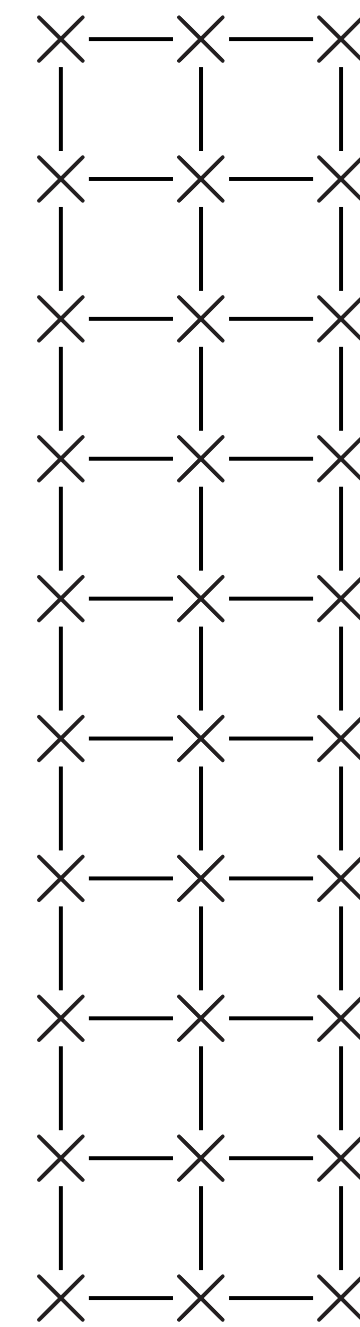

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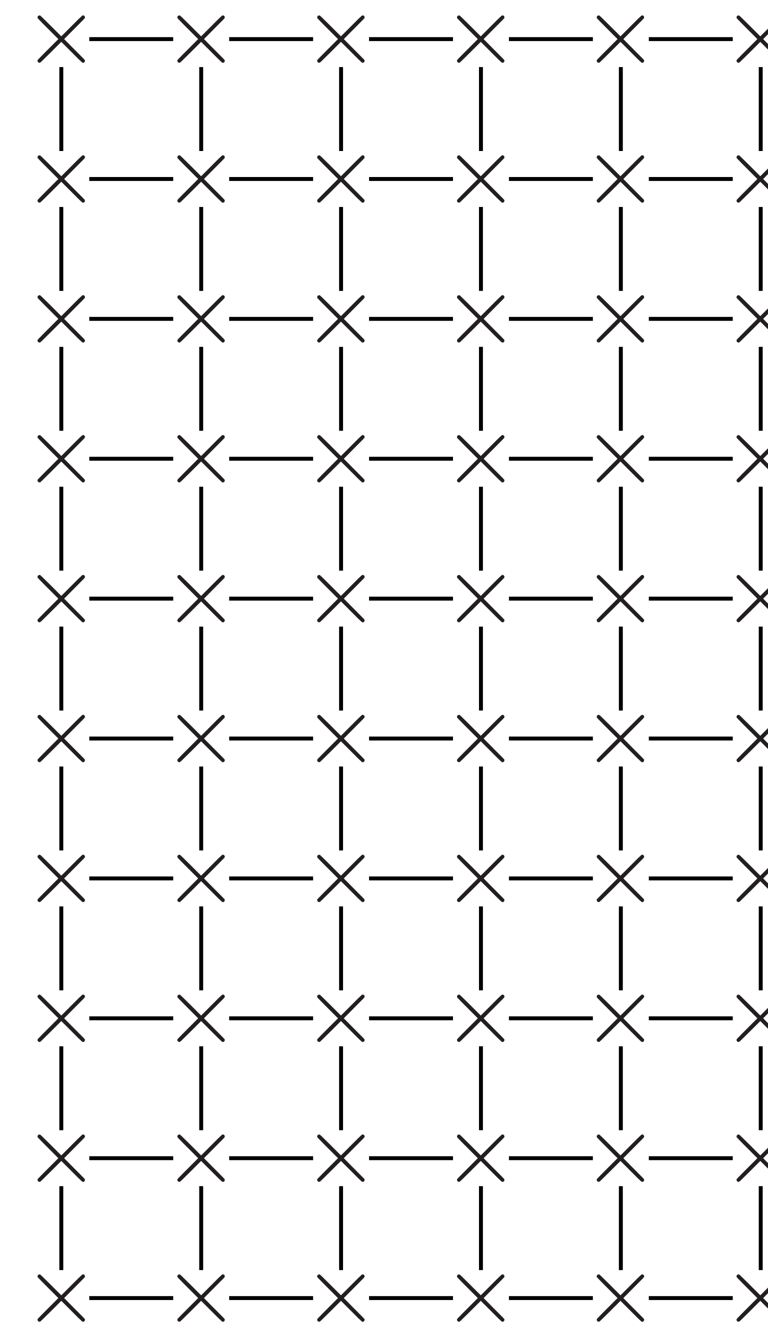

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Spread array across
square mesh of $n$ small cells
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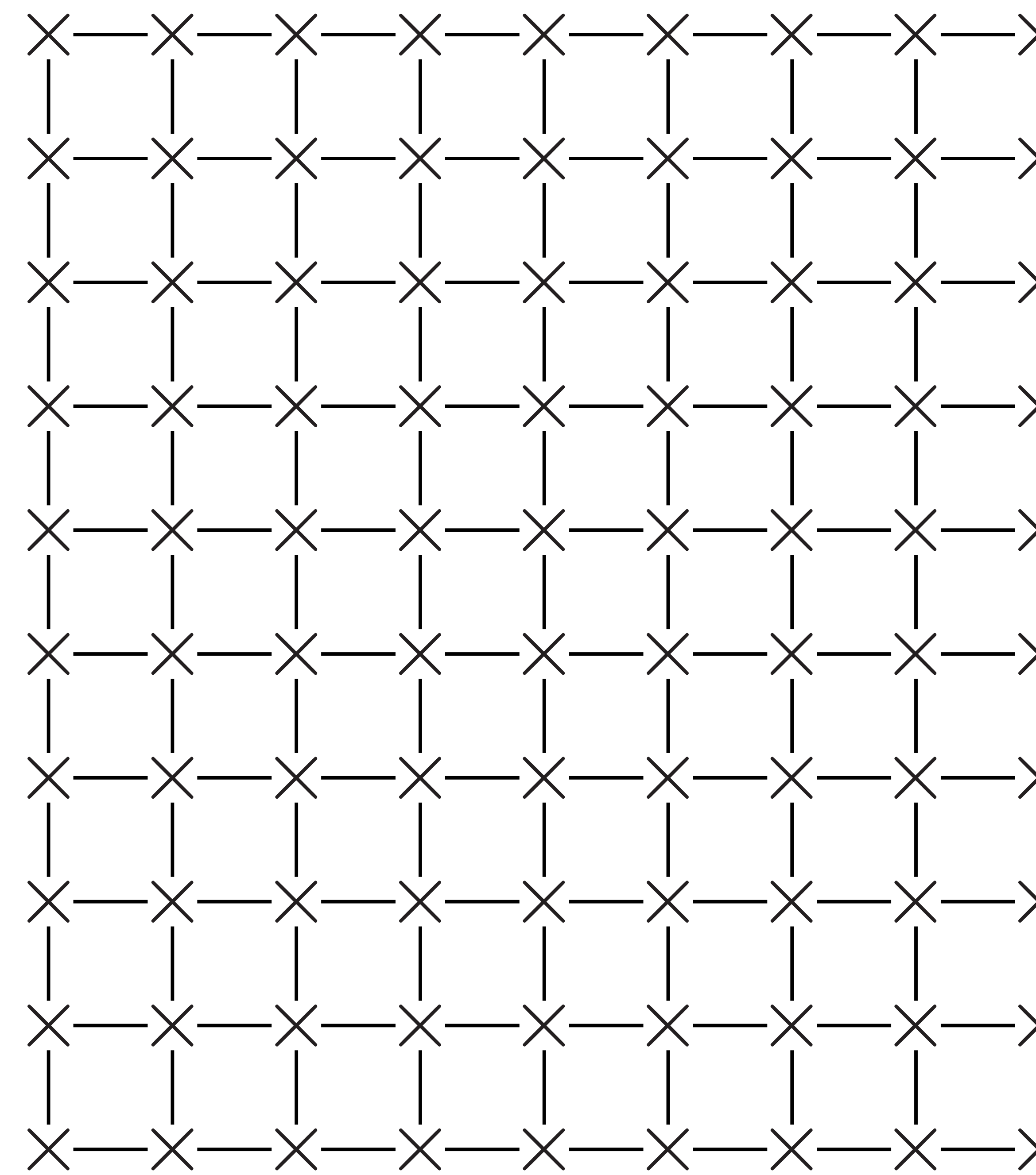
with near-neighbor wiring:



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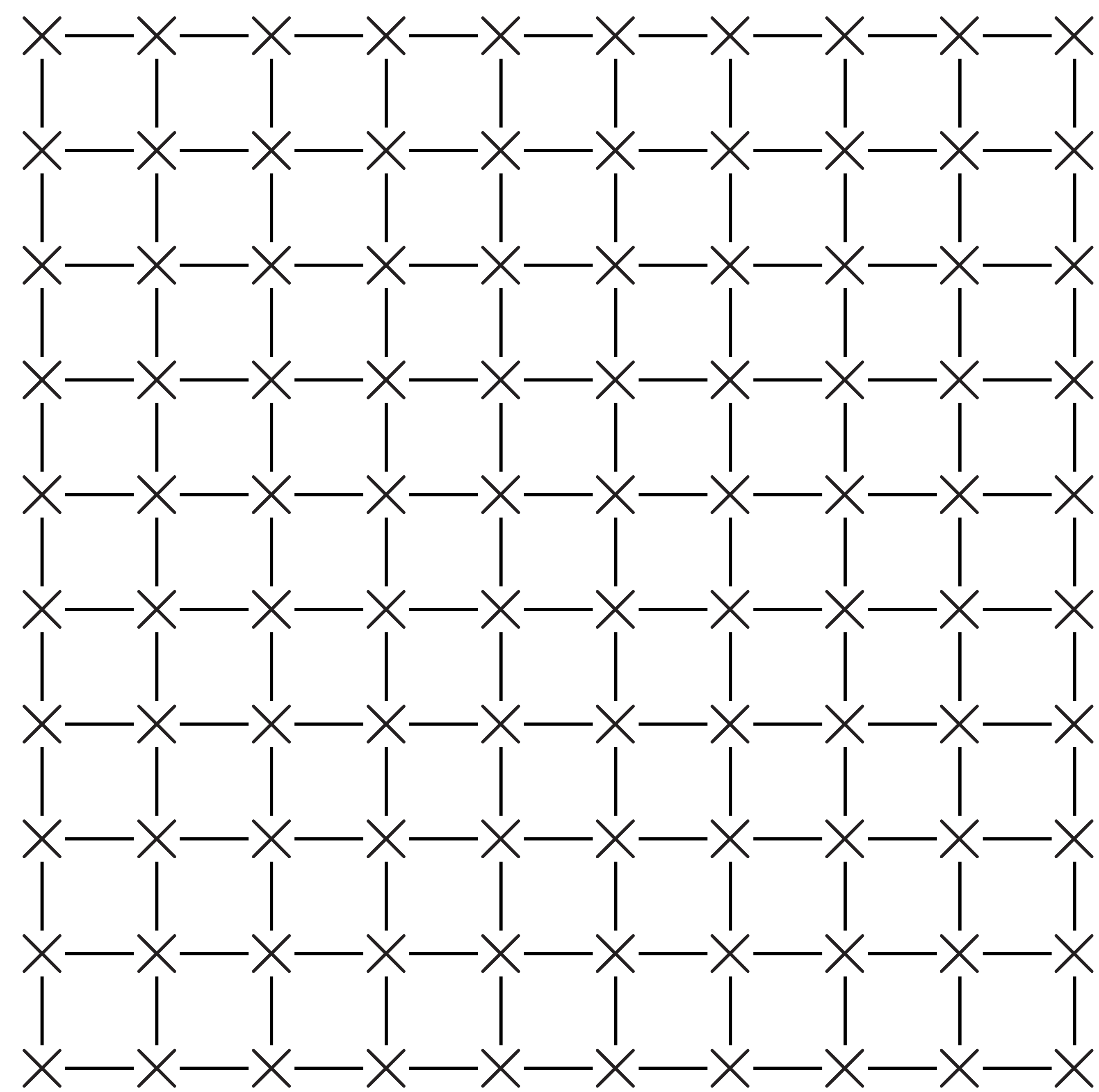

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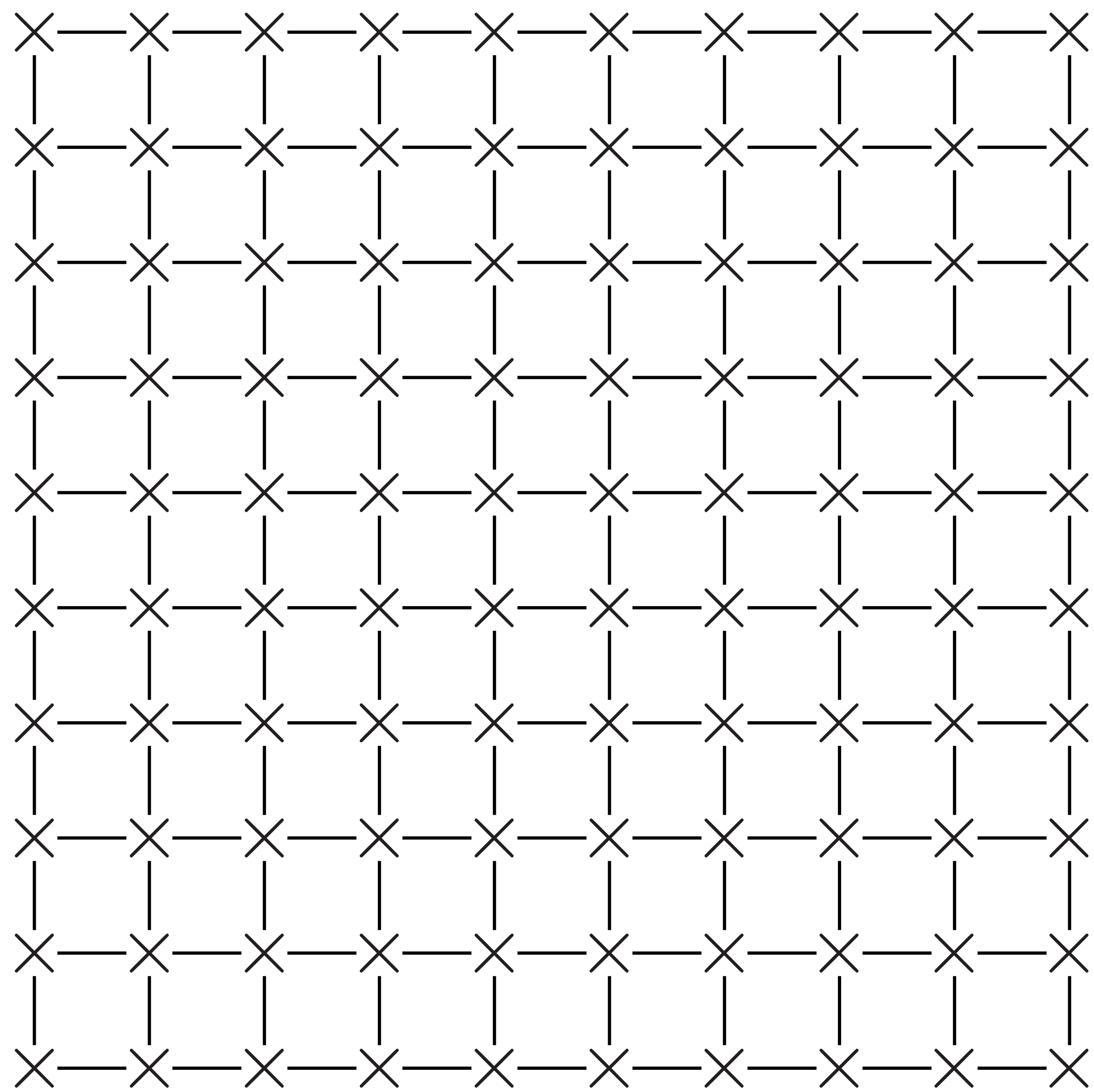

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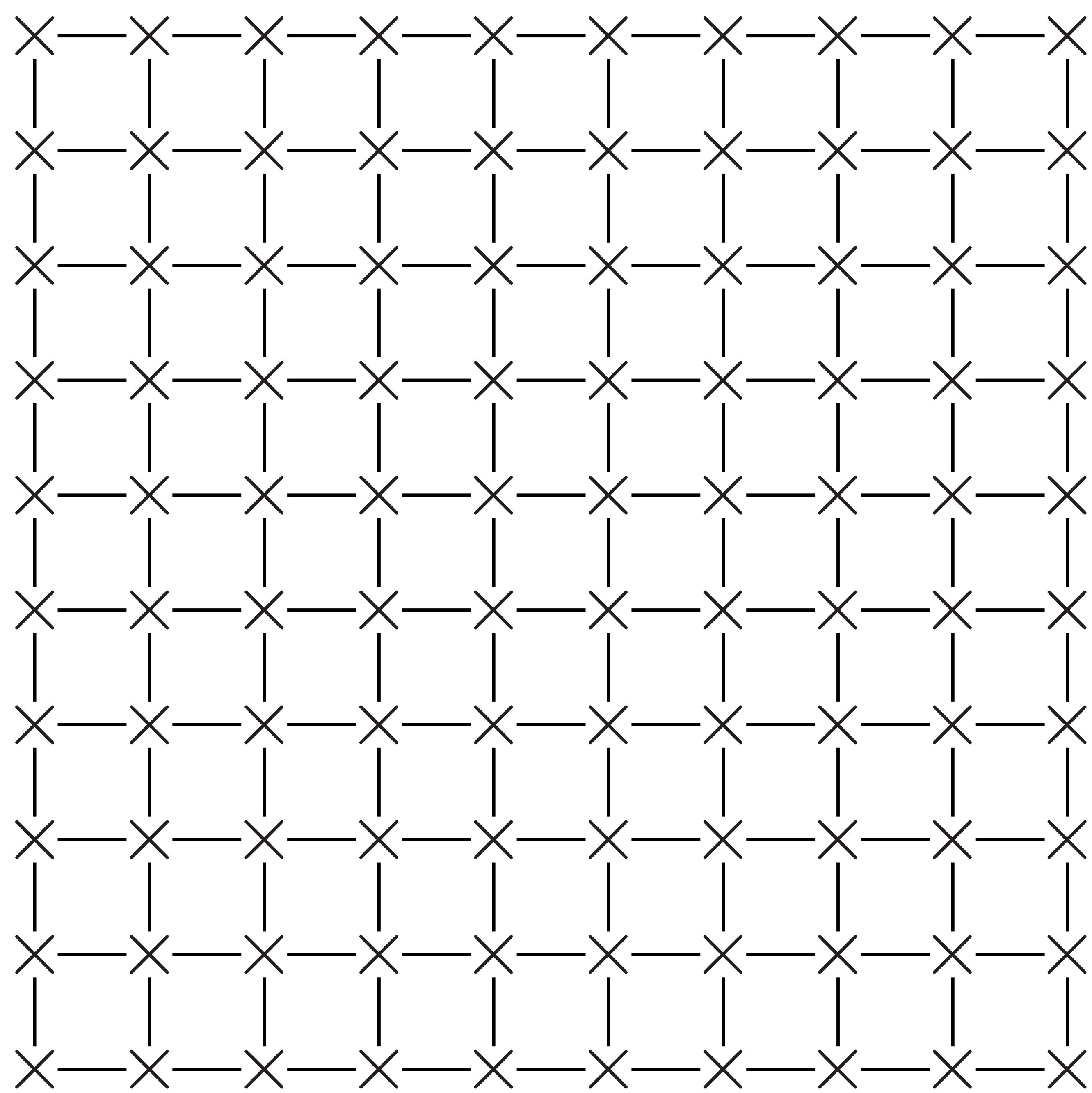

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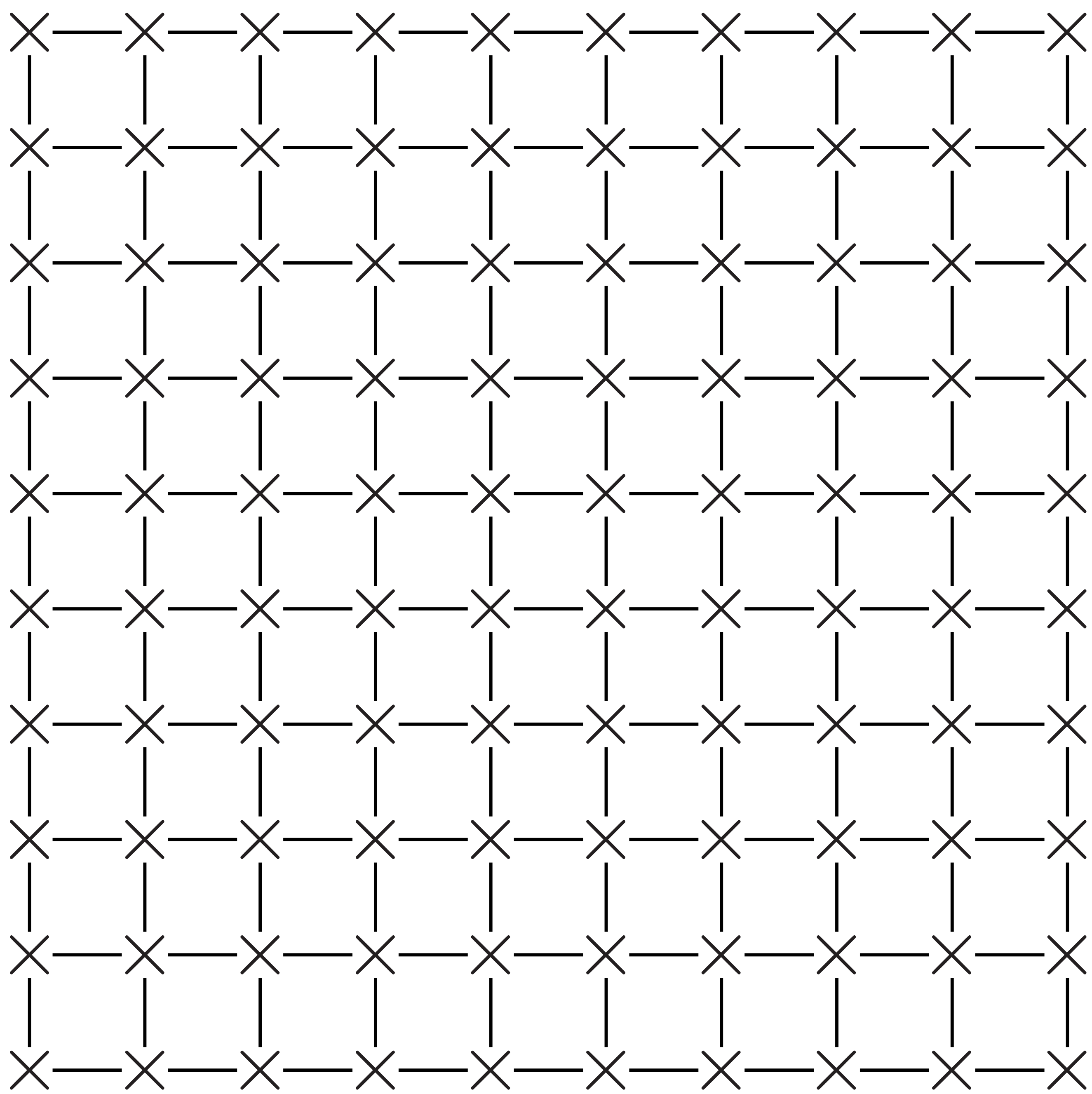



Sort row of $n^{0.5}$ cells
in $n^{0.5+o(1)}$ seconds:

- Sort each pair in parallel.
  $\underline{3\ 1}\ \underline{4\ 1}\ \underline{5\ 9}\ \underline{2\ 6} \mapsto$
  $1\ 3\ 1\ 4\ 5\ 9\ 2\ 6$

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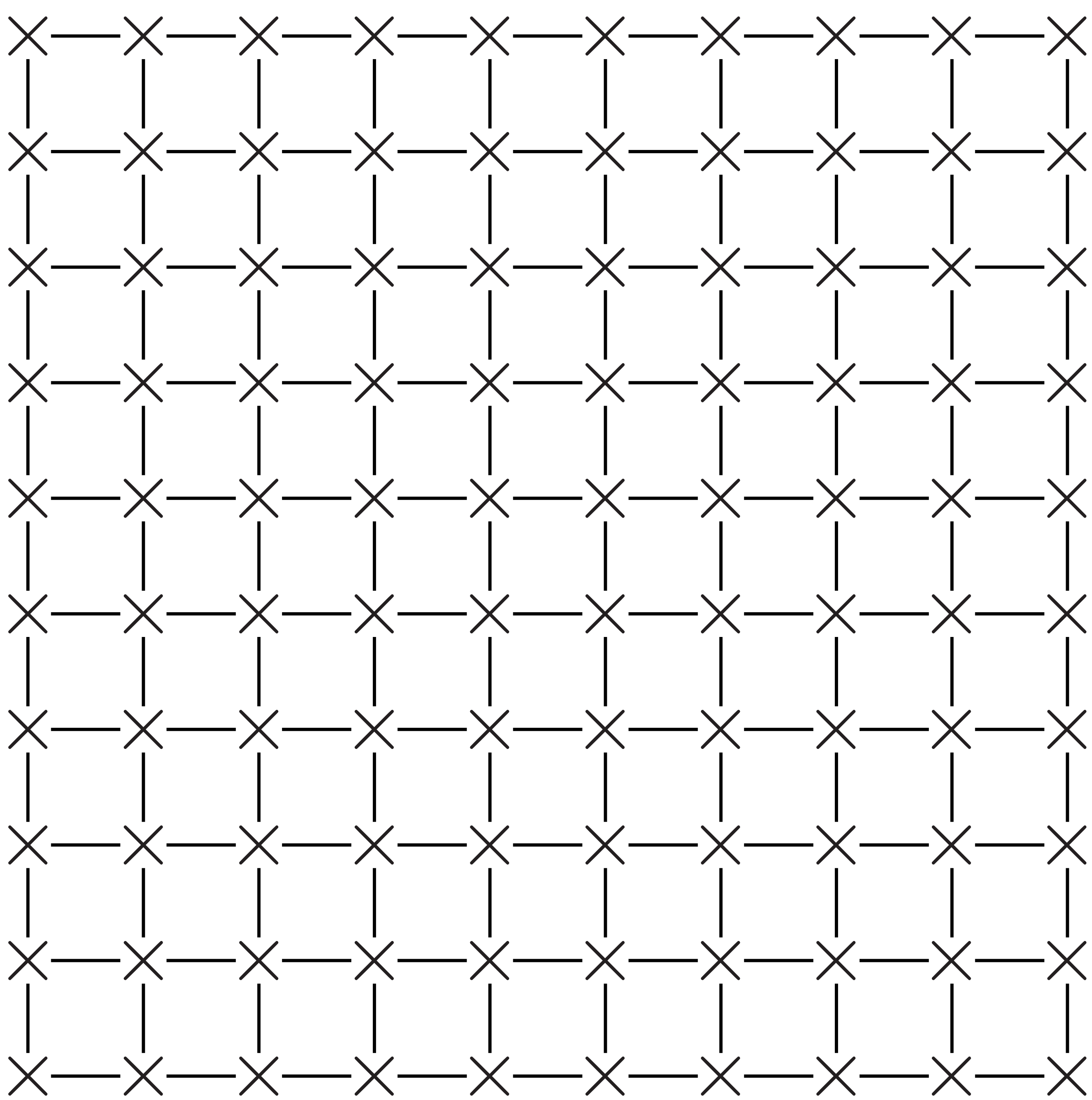



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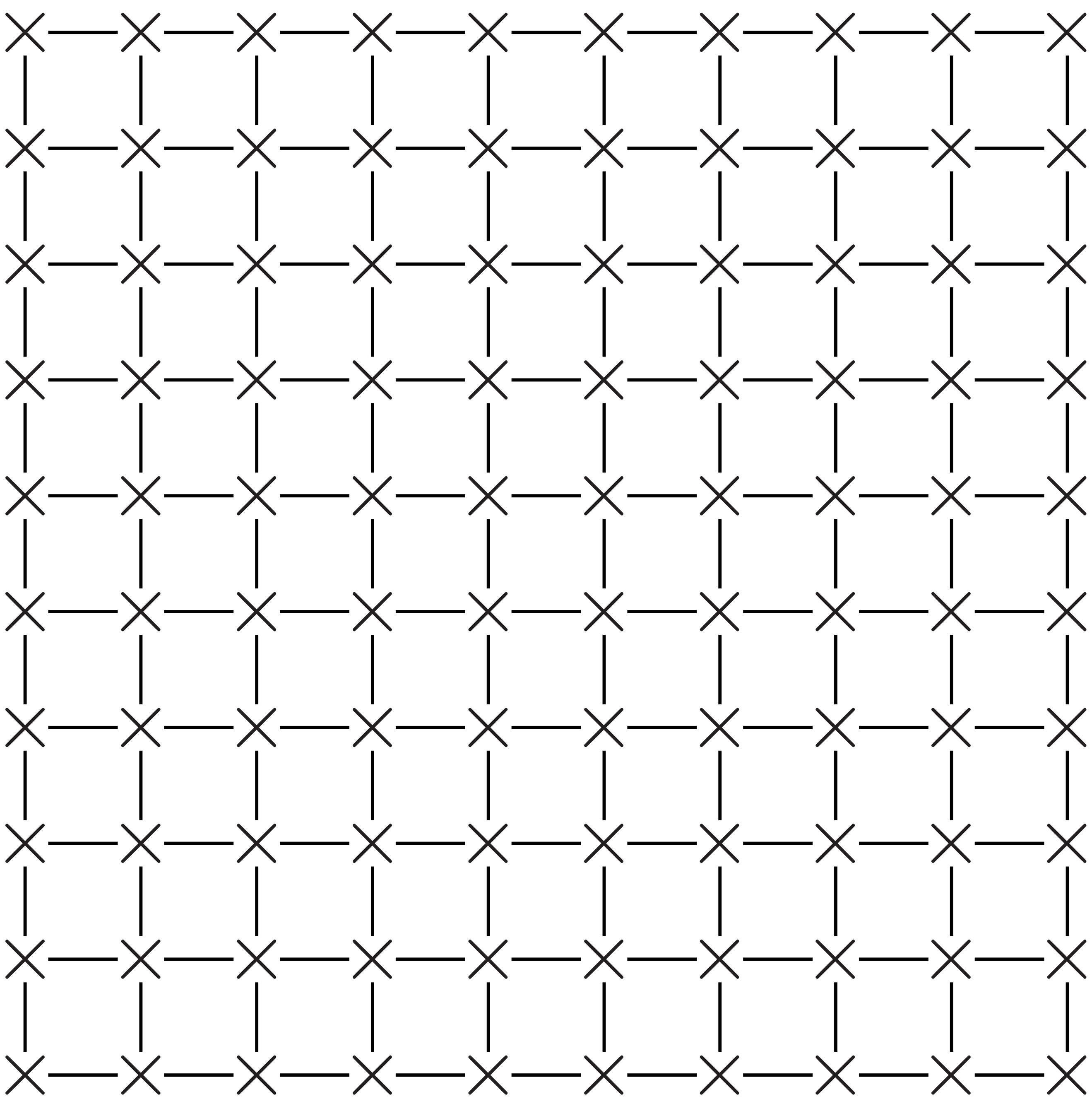



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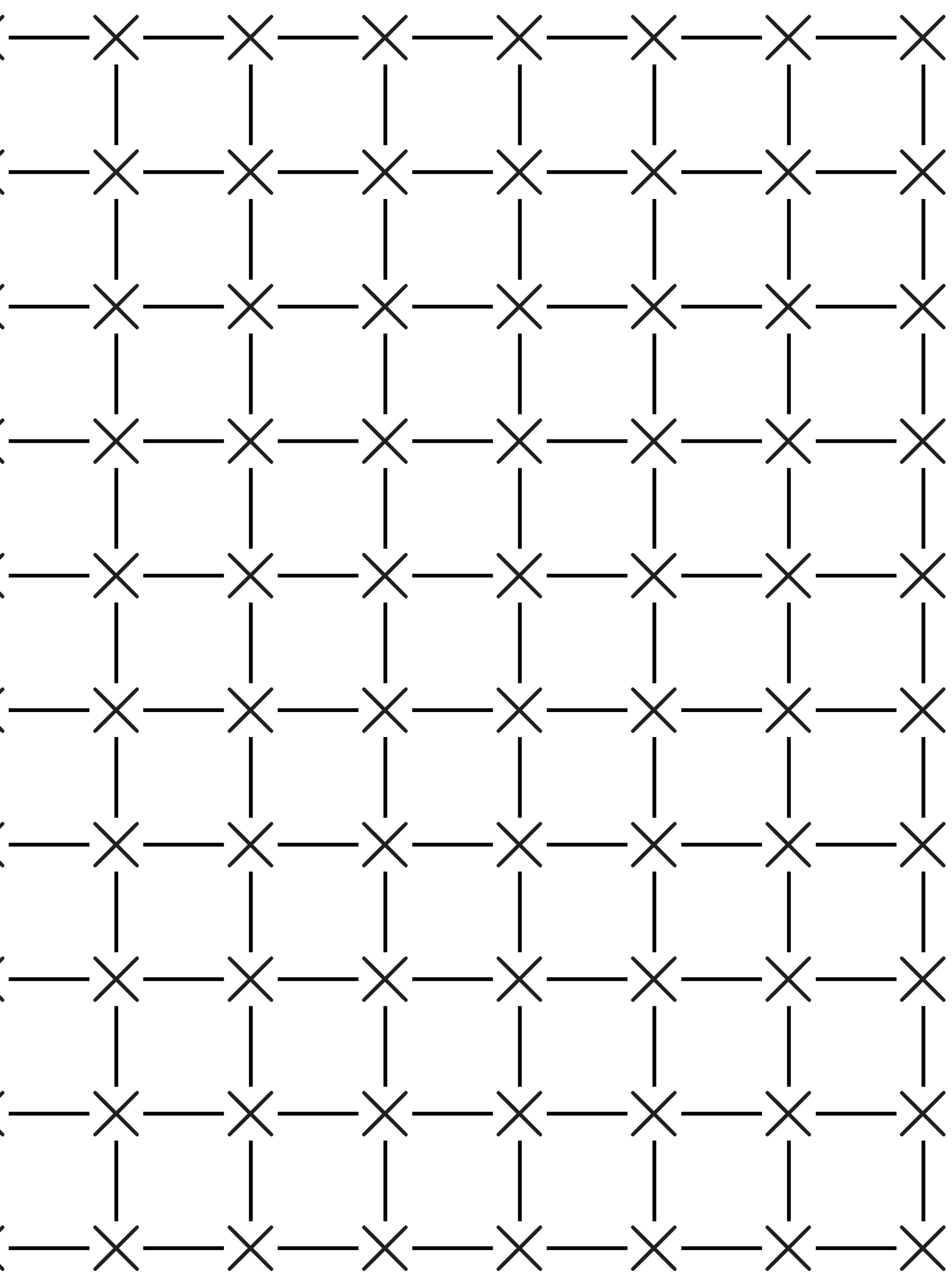



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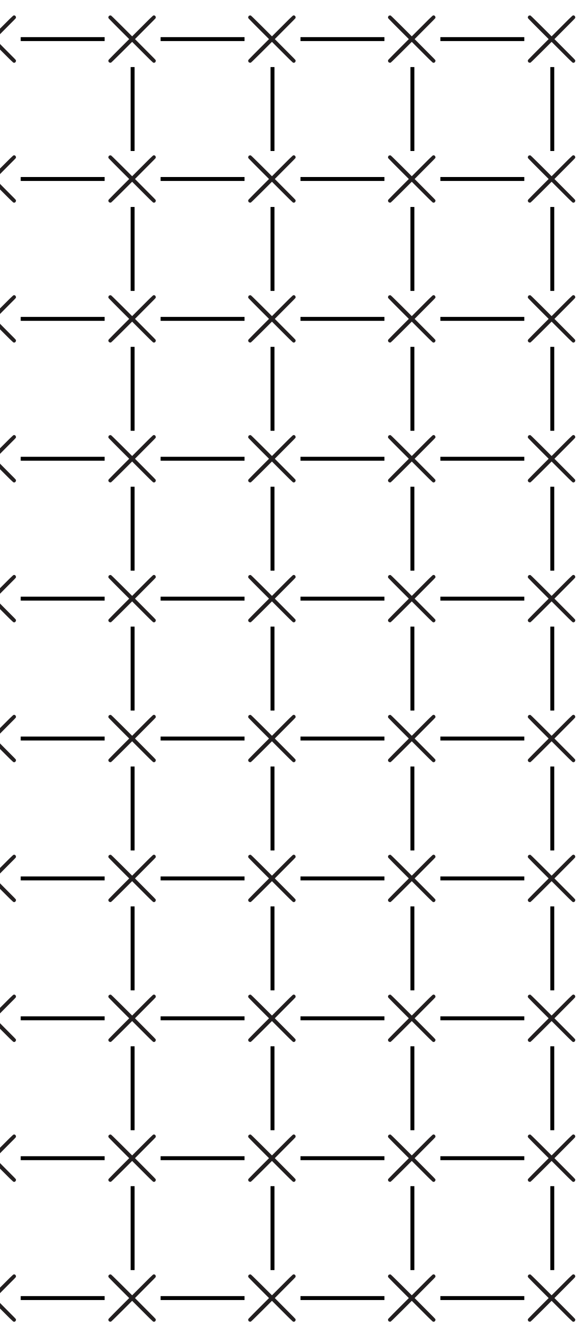



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Sort all $n$ cells
in $n^{0.5+o(1)}$ secon

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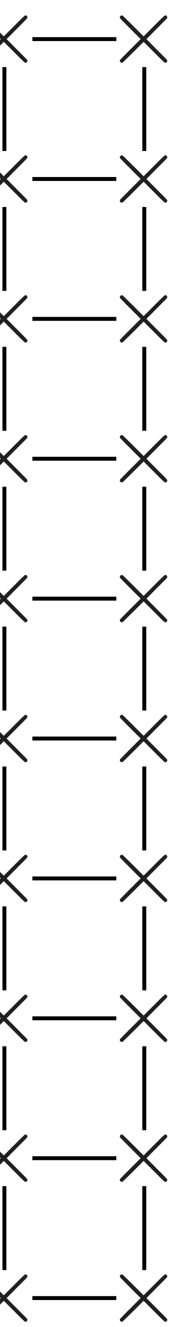



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Sort all $n$ cells
in $n^{0.5+o(1)}$ seconds:

- Recursively sort quadrants in parallel, if $n > 1$.
- Sort each column in parall
- Sort each row in parallel.
- Sort each column in parall
- Sort each row in parallel.

With proper choice of
left-to-right/right-to-left
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in a *total* of $n^{0.5+o(1)}$ seconds.



Sort all $n$ cells
in $n^{0.5+o(1)}$ seconds:

- Recursively sort quadrants in parallel, if $n > 1$.
- Sort each column in parallel.
- Sort each row in parallel.
- Sort each column in parallel.
- Sort each row in parallel.

With proper choice of
left-to-right/right-to-left
for each row, can prove
that this sorts whole array.

/ of $n^{0.5}$ cells
$^{o(1)}$ seconds:

ach pair in parallel.

1 5 9 2 6 $\mapsto$

4 5 9 2 6

lternate pairs in parallel.

4 5 9 2 6 $\mapsto$

4 5 2 9 6

t until number of steps
row length.

h row, in parallel,
al of $n^{0.5+o(1)}$ seconds.



Sort all $n$ cells
in $n^{0.5+o(1)}$ seconds:

- Recursively sort quadrants in parallel, if $n > 1$.
- Sort each column in parallel.
- Sort each row in parallel.
- Sort each column in parallel.
- Sort each row in parallel.

With proper choice of
left-to-right/right-to-left
for each row, can prove
that this sorts whole array.

For exan
this $8 \times$

| | | |
|---|---|---|
| 3 | 1 | 4 |
| 5 | 3 | 5 |
| 2 | 3 | 8 |
| 3 | 3 | 8 |
| 0 | 2 | 8 |
| 1 | 6 | 9 |
| 5 | 1 | 0 |
| 7 | 4 | 9 |

ells

ds:

n parallel.

$\mapsto$

airs in parallel.

$\mapsto$

mber of steps

ch.

parallel,

$o(1)$ seconds.



Sort all $n$ cells
in $n^{0.5+o(1)}$ seconds:

- Recursively sort quadrants in parallel, if $n > 1$.
- Sort each column in parallel.
- Sort each row in parallel.
- Sort each column in parallel.
- Sort each row in parallel.

With proper choice of
left-to-right/right-to-left
for each row, can prove
that this sorts whole array.

For example, assu
this $8 \times 8$ array is

| | | | | | |
|---|---|---|---|---|---|
| 3 | 1 | 4 | 1 | 5 | 9 |
| 5 | 3 | 5 | 8 | 9 | 7 |
| 2 | 3 | 8 | 4 | 6 | 2 |
| 3 | 3 | 8 | 3 | 2 | 7 |
| 0 | 2 | 8 | 8 | 4 | 1 |
| 1 | 6 | 9 | 3 | 9 | 9 |
| 5 | 1 | 0 | 5 | 8 | 2 |
| 7 | 4 | 9 | 4 | 4 | 5 |

allel.

eps

ds.



Sort all $n$ cells
in $n^{0.5+o(1)}$ seconds:

- Recursively sort quadrants in parallel, if $n > 1$.
- Sort each column in parallel.
- Sort each row in parallel.
- Sort each column in parallel.
- Sort each row in parallel.

With proper choice of
left-to-right/right-to-left
for each row, can prove
that this sorts whole array.



For example, assume that
this $8 \times 8$ array is in cells:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 1 | 4 | 1 | 5 | 9 | 2 | 6 |
| 5 | 3 | 5 | 8 | 9 | 7 | 9 | 3 |
| 2 | 3 | 8 | 4 | 6 | 2 | 6 | 4 |
| 3 | 3 | 8 | 3 | 2 | 7 | 9 | 5 |
| 0 | 2 | 8 | 8 | 4 | 1 | 9 | 7 |
| 1 | 6 | 9 | 3 | 9 | 9 | 3 | 7 |
| 5 | 1 | 0 | 5 | 8 | 2 | 0 | 9 |
| 7 | 4 | 9 | 4 | 4 | 5 | 9 | 2 |

Sort all $n$ cells
in $n^{0.5+o(1)}$ seconds:

- Recursively sort quadrants in parallel, if $n > 1$.
- Sort each column in parallel.
- Sort each row in parallel.
- Sort each column in parallel.
- Sort each row in parallel.

With proper choice of
left-to-right/right-to-left
for each row, can prove
that this sorts whole array.



For example, assume that
this $8 \times 8$ array is in cells:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 1 | 4 | 1 | 5 | 9 | 2 | 6 |
| 5 | 3 | 5 | 8 | 9 | 7 | 9 | 3 |
| 2 | 3 | 8 | 4 | 6 | 2 | 6 | 4 |
| 3 | 3 | 8 | 3 | 2 | 7 | 9 | 5 |
| 0 | 2 | 8 | 8 | 4 | 1 | 9 | 7 |
| 1 | 6 | 9 | 3 | 9 | 9 | 3 | 7 |
| 5 | 1 | 0 | 5 | 8 | 2 | 0 | 9 |
| 7 | 4 | 9 | 4 | 4 | 5 | 9 | 2 |

n cells
o(1) seconds:

sively sort quadrants
allel, if $n > 1$.
ach column in parallel.
ach row in parallel.
ach column in parallel.
ach row in parallel.

oper choice of
ght/right-to-left
row, can prove
s sorts whole array.



For example, assume that this $8 \times 8$ array is in cells:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 1 | 4 | 1 | 5 | 9 | 2 | 6 |
| 5 | 3 | 5 | 8 | 9 | 7 | 9 | 3 |
| 2 | 3 | 8 | 4 | 6 | 2 | 6 | 4 |
| 3 | 3 | 8 | 3 | 2 | 7 | 9 | 5 |
| 0 | 2 | 8 | 8 | 4 | 1 | 9 | 7 |
| 1 | 6 | 9 | 3 | 9 | 9 | 3 | 7 |
| 5 | 1 | 0 | 5 | 8 | 2 | 0 | 9 |
| 7 | 4 | 9 | 4 | 4 | 5 | 9 | 2 |

Recursiv
top →,

| | | |
|---|---|---|
| 1 | 1 | 2 |
| 3 | 3 | 3 |
| 3 | 4 | 4 |
| 5 | 8 | 8 |
| 1 | 1 | 0 |
| 4 | 4 | 3 |
| 7 | 6 | 5 |
| 9 | 9 | 8 |

ds:

quadrants

> 1.

n in parallel.

n parallel.

n in parallel.

n parallel.

e of

to-left

prove

ole array.



For example, assume that this $8 \times 8$ array is in cells:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 1 | 4 | 1 | 5 | 9 | 2 | 6 |
| 5 | 3 | 5 | 8 | 9 | 7 | 9 | 3 |
| 2 | 3 | 8 | 4 | 6 | 2 | 6 | 4 |
| 3 | 3 | 8 | 3 | 2 | 7 | 9 | 5 |
| 0 | 2 | 8 | 8 | 4 | 1 | 9 | 7 |
| 1 | 6 | 9 | 3 | 9 | 9 | 3 | 7 |
| 5 | 1 | 0 | 5 | 8 | 2 | 0 | 9 |
| 7 | 4 | 9 | 4 | 4 | 5 | 9 | 2 |

Recursively sort qu

top $\rightarrow$, bottom $\leftarrow$

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 2 | 2 |
| 3 | 3 | 3 | 3 | 4 | 5 |
| 3 | 4 | 4 | 5 | 6 | 6 |
| 5 | 8 | 8 | 8 | 9 | 9 |
| 1 | 1 | 0 | 0 | 2 | 2 |
| 4 | 4 | 3 | 2 | 5 | 4 |
| 7 | 6 | 5 | 5 | 9 | 8 |
| 9 | 9 | 8 | 8 | 9 | 9 |

el.

el.



For example, assume that this $8 \times 8$ array is in cells:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 1 | 4 | 1 | 5 | 9 | 2 | 6 |
| 5 | 3 | 5 | 8 | 9 | 7 | 9 | 3 |
| 2 | 3 | 8 | 4 | 6 | 2 | 6 | 4 |
| 3 | 3 | 8 | 3 | 2 | 7 | 9 | 5 |
| 0 | 2 | 8 | 8 | 4 | 1 | 9 | 7 |
| 1 | 6 | 9 | 3 | 9 | 9 | 3 | 7 |
| 5 | 1 | 0 | 5 | 8 | 2 | 0 | 9 |
| 7 | 4 | 9 | 4 | 4 | 5 | 9 | 2 |



Recursively sort quadrants, top $\rightarrow$, bottom $\leftarrow$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 4 | 5 | 5 | 6 |
| 3 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 5 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 1 | 1 | 0 | 0 | 2 | 2 | 1 | 0 |
| 4 | 4 | 3 | 2 | 5 | 4 | 4 | 3 |
| 7 | 6 | 5 | 5 | 9 | 8 | 7 | 7 |
| 9 | 9 | 8 | 8 | 9 | 9 | 9 | 9 |

For example, assume that this $8 \times 8$ array is in cells:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 1 | 4 | 1 | 5 | 9 | 2 | 6 |
| 5 | 3 | 5 | 8 | 9 | 7 | 9 | 3 |
| 2 | 3 | 8 | 4 | 6 | 2 | 6 | 4 |
| 3 | 3 | 8 | 3 | 2 | 7 | 9 | 5 |
| 0 | 2 | 8 | 8 | 4 | 1 | 9 | 7 |
| 1 | 6 | 9 | 3 | 9 | 9 | 3 | 7 |
| 5 | 1 | 0 | 5 | 8 | 2 | 0 | 9 |
| 7 | 4 | 9 | 4 | 4 | 5 | 9 | 2 |



Recursively sort quadrants, top $\rightarrow$, bottom $\leftarrow$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 4 | 5 | 5 | 6 |
| 3 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 5 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 1 | 1 | 0 | 0 | 2 | 2 | 1 | 0 |
| 4 | 4 | 3 | 2 | 5 | 4 | 4 | 3 |
| 7 | 6 | 5 | 5 | 9 | 8 | 7 | 7 |
| 9 | 9 | 8 | 8 | 9 | 9 | 9 | 9 |

mple, assume that

8 array is in cells:

| | | | | | |
|---|---|---|---|---|---|
| 4 | 1 | 5 | 9 | 2 | 6 |
| 5 | 8 | 9 | 7 | 9 | 3 |
| 3 | 4 | 6 | 2 | 6 | 4 |
| 3 | 3 | 2 | 7 | 9 | 5 |
| 3 | 8 | 4 | 1 | 9 | 7 |
| 9 | 3 | 9 | 9 | 3 | 7 |
| 0 | 5 | 8 | 2 | 0 | 9 |
| 9 | 4 | 4 | 5 | 9 | 2 |



Recursively sort quadrants,

top $\rightarrow$, bottom $\leftarrow$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 4 | 5 | 5 | 6 |
| 3 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 5 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 1 | 1 | 0 | 0 | 2 | 2 | 1 | 0 |
| 4 | 4 | 3 | 2 | 5 | 4 | 4 | 3 |
| 7 | 6 | 5 | 5 | 9 | 8 | 7 | 7 |
| 9 | 9 | 8 | 8 | 9 | 9 | 9 | 9 |

Sort eac

in parall

| | | |
|---|---|---|
| 1 | 1 | 0 |
| 1 | 1 | 2 |
| 3 | 3 | 3 |
| 3 | 4 | 3 |
| 4 | 4 | 4 |
| 5 | 6 | 5 |
| 7 | 8 | 8 |
| 9 | 9 | 8 |

me that
in cells:

| | |
|---|---|
| 2 | 6 |
| 9 | 3 |
| 6 | 4 |
| 9 | 5 |
| 9 | 7 |
| 3 | 7 |
| 0 | 9 |
| 9 | 2 |





Recursively sort quadrants,
top $\rightarrow$, bottom $\leftarrow$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 4 | 5 | 5 | 6 |
| 3 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 5 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 1 | 1 | 0 | 0 | 2 | 2 | 1 | 0 |
| 4 | 4 | 3 | 2 | 5 | 4 | 4 | 3 |
| 7 | 6 | 5 | 5 | 9 | 8 | 7 | 7 |
| 9 | 9 | 8 | 8 | 9 | 9 | 9 | 9 |

Sort each column
in parallel:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 2 | 2 |
| 1 | 1 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 4 | 4 |
| 3 | 4 | 3 | 3 | 5 | 5 |
| 4 | 4 | 4 | 5 | 6 | 6 |
| 5 | 6 | 5 | 5 | 9 | 8 |
| 7 | 8 | 8 | 8 | 9 | 9 |
| 9 | 9 | 8 | 8 | 9 | 9 |

Recursively sort quadrants,
top →, bottom ←:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 4 | 5 | 5 | 6 |
| 3 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 5 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 1 | 1 | 0 | 0 | 2 | 2 | 1 | 0 |
| 4 | 4 | 3 | 2 | 5 | 4 | 4 | 3 |
| 7 | 6 | 5 | 5 | 9 | 8 | 7 | 7 |
| 9 | 9 | 8 | 8 | 9 | 9 | 9 | 9 |



Sort each column
in parallel:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 2 | 2 | 1 | 0 |
| 1 | 1 | 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 4 | 4 | 4 | 3 |
| 3 | 4 | 3 | 3 | 5 | 5 | 5 | 6 |
| 4 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 5 | 6 | 5 | 5 | 9 | 8 | 7 | 7 |
| 7 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 9 | 9 | 8 | 8 | 9 | 9 | 9 | 9 |

Recursively sort quadrants,

top $\rightarrow$, bottom $\leftarrow$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 4 | 5 | 5 | 6 |
| 3 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 5 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 1 | 1 | 0 | 0 | 2 | 2 | 1 | 0 |
| 4 | 4 | 3 | 2 | 5 | 4 | 4 | 3 |
| 7 | 6 | 5 | 5 | 9 | 8 | 7 | 7 |
| 9 | 9 | 8 | 8 | 9 | 9 | 9 | 9 |



Sort each column

in parallel:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 2 | 2 | 1 | 0 |
| 1 | 1 | 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 4 | 4 | 4 | 3 |
| 3 | 4 | 3 | 3 | 5 | 5 | 5 | 6 |
| 4 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 5 | 6 | 5 | 5 | 9 | 8 | 7 | 7 |
| 7 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 9 | 9 | 8 | 8 | 9 | 9 | 9 | 9 |

ely sort quadrants,

bottom ←:

| | | | | | |
|---|---|---|---|---|---|
| 2 | 3 | 2 | 2 | 2 | 3 |
| 3 | 3 | 4 | 5 | 5 | 6 |
| 4 | 5 | 6 | 6 | 7 | 7 |
| 3 | 8 | 9 | 9 | 9 | 9 |
| 0 | 0 | 2 | 2 | 1 | 0 |
| 3 | 2 | 5 | 4 | 4 | 3 |
| 5 | 5 | 9 | 8 | 7 | 7 |
| 3 | 8 | 9 | 9 | 9 | 9 |



Sort each column

in parallel:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 2 | 2 | 1 | 0 |
| 1 | 1 | 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 4 | 4 | 4 | 3 |
| 3 | 4 | 3 | 3 | 5 | 5 | 5 | 6 |
| 4 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 5 | 6 | 5 | 5 | 9 | 8 | 7 | 7 |
| 7 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 9 | 9 | 8 | 8 | 9 | 9 | 9 | 9 |

Sort eac

alternate

| | | |
|---|---|---|
| 0 | 0 | 0 |
| 3 | 2 | 2 |
| 3 | 3 | 3 |
| 6 | 5 | 5 |
| 4 | 4 | 4 |
| 9 | 8 | 7 |
| 7 | 8 | 8 |
| 9 | 9 | 9 |

uadrants,

:

| | |
|---|---|
| 2 | 3 |
| 5 | 6 |
| 7 | 7 |
| 9 | 9 |
| 1 | 0 |
| 4 | 3 |
| 7 | 7 |
| 9 | 9 |



Sort each column
in parallel:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 2 | 2 | 1 | 0 |
| 1 | 1 | 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 4 | 4 | 4 | 3 |
| 3 | 4 | 3 | 3 | 5 | 5 | 5 | 6 |
| 4 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 5 | 6 | 5 | 5 | 9 | 8 | 7 | 7 |
| 7 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 9 | 9 | 8 | 8 | 9 | 9 | 9 | 9 |

Sort each row in p
alternately $\leftarrow$, $\rightarrow$:

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 |
| 3 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 4 |
| 6 | 5 | 5 | 5 | 4 | 3 |
| 4 | 4 | 4 | 5 | 6 | 6 |
| 9 | 8 | 7 | 7 | 6 | 5 |
| 7 | 8 | 8 | 8 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 |

Sort each column
in parallel:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 2 | 2 | 1 | 0 |
| 1 | 1 | 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 4 | 4 | 4 | 3 |
| 3 | 4 | 3 | 3 | 5 | 5 | 5 | 6 |
| 4 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 5 | 6 | 5 | 5 | 9 | 8 | 7 | 7 |
| 7 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 9 | 9 | 8 | 8 | 9 | 9 | 9 | 9 |



Sort each row in parallel,
alternately $\leftarrow$, $\rightarrow$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 |
| 3 | 2 | 2 | 2 | 2 | 2 | 1 | 1 |
| 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 |
| 6 | 5 | 5 | 5 | 4 | 3 | 3 | 3 |
| 4 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 9 | 8 | 7 | 7 | 6 | 5 | 5 | 5 |
| 7 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 8 | 8 |

Sort each column
in parallel:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 2 | 2 | 1 | 0 |
| 1 | 1 | 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 4 | 4 | 4 | 3 |
| 3 | 4 | 3 | 3 | 5 | 5 | 5 | 6 |
| 4 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 5 | 6 | 5 | 5 | 9 | 8 | 7 | 7 |
| 7 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 9 | 9 | 8 | 8 | 9 | 9 | 9 | 9 |



Sort each row in parallel,
alternately $\leftarrow$, $\rightarrow$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 |
| 3 | 2 | 2 | 2 | 2 | 2 | 1 | 1 |
| 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 |
| 6 | 5 | 5 | 5 | 4 | 3 | 3 | 3 |
| 4 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 9 | 8 | 7 | 7 | 6 | 5 | 5 | 5 |
| 7 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 8 | 8 |

h column

el:

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 2 | 2 | 1 | 0 |
| 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 4 | 4 | 4 | 3 |
| 3 | 3 | 5 | 5 | 5 | 6 |
| 4 | 5 | 6 | 6 | 7 | 7 |
| 5 | 5 | 9 | 8 | 7 | 7 |
| 8 | 8 | 9 | 9 | 9 | 9 |
| 8 | 8 | 9 | 9 | 9 | 9 |



Sort each row in parallel,

alternately $\leftarrow$, $\rightarrow$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 |
| 3 | 2 | 2 | 2 | 2 | 2 | 1 | 1 |
| 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 |
| 6 | 5 | 5 | 5 | 4 | 3 | 3 | 3 |
| 4 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 9 | 8 | 7 | 7 | 6 | 5 | 5 | 5 |
| 7 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 8 | 8 |

Sort eac

in parall

| | | |
|---|---|---|
| 0 | 0 | 0 |
| 3 | 2 | 2 |
| 3 | 3 | 3 |
| 4 | 4 | 4 |
| 6 | 5 | 5 |
| 7 | 8 | 7 |
| 9 | 8 | 8 |
| 9 | 9 | 9 |

| | |
|---|---|
| 1 | 0 |
| 2 | 3 |
| 4 | 3 |
| 5 | 6 |
| 7 | 7 |
| 7 | 7 |
| 9 | 9 |
| 9 | 9 |



Sort each row in parallel, alternately $\leftarrow$, $\rightarrow$:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 |
| 3 | 2 | 2 | 2 | 2 | 2 | 1 | 1 |
| 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 |
| 6 | 5 | 5 | 5 | 4 | 3 | 3 | 3 |
| 4 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 9 | 8 | 7 | 7 | 6 | 5 | 5 | 5 |
| 7 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 8 | 8 |

Sort each column in parallel:

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 |
| 3 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 5 | 4 | 4 |
| 6 | 5 | 5 | 5 | 6 | 5 |
| 7 | 8 | 7 | 7 | 6 | 6 |
| 9 | 8 | 8 | 8 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 |

Sort each row in parallel, alternately ←, →:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 |
| 3 | 2 | 2 | 2 | 2 | 2 | 1 | 1 |
| 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 |
| 6 | 5 | 5 | 5 | 4 | 3 | 3 | 3 |
| 4 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 9 | 8 | 7 | 7 | 6 | 5 | 5 | 5 |
| 7 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 8 | 8 |



Sort each column in parallel:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 5 | 4 | 4 | 4 | 4 |
| 6 | 5 | 5 | 5 | 6 | 5 | 5 | 5 |
| 7 | 8 | 7 | 7 | 6 | 6 | 7 | 7 |
| 9 | 8 | 8 | 8 | 9 | 9 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

Sort each row in parallel,

alternately ←, →:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 |
| 3 | 2 | 2 | 2 | 2 | 2 | 1 | 1 |
| 3 | 3 | 3 | 3 | 3 | 4 | 4 | 4 |
| 6 | 5 | 5 | 5 | 4 | 3 | 3 | 3 |
| 4 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 9 | 8 | 7 | 7 | 6 | 5 | 5 | 5 |
| 7 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 8 | 8 |



Sort each column

in parallel:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 5 | 4 | 4 | 4 | 4 |
| 6 | 5 | 5 | 5 | 6 | 5 | 5 | 5 |
| 7 | 8 | 7 | 7 | 6 | 6 | 7 | 7 |
| 9 | 8 | 8 | 8 | 9 | 9 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

h row in parallel,

ely ←, →:

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 2 |
| 2 | 2 | 2 | 1 | 1 |
| 3 | 3 | 4 | 4 | 4 |
| 5 | 4 | 3 | 3 | 3 |
| 5 | 6 | 6 | 7 | 7 |
| 7 | 6 | 5 | 5 | 5 |
| 8 | 9 | 9 | 9 | 9 |
| 9 | 9 | 9 | 8 | 8 |



Sort each column

in parallel:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 5 | 4 | 4 | 4 | 4 |
| 6 | 5 | 5 | 5 | 6 | 5 | 5 | 5 |
| 7 | 8 | 7 | 7 | 6 | 6 | 7 | 7 |
| 9 | 8 | 8 | 8 | 9 | 9 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

Sort eac

← or →

| | |
|---|---|
| 0 | 0 |
| 2 | 2 |
| 3 | 3 |
| 4 | 4 |
| 5 | 5 |
| 6 | 6 |
| 8 | 8 |
| 9 | 9 |

parallel,

| | |
|---|---|
| 2 | 2 |
| 1 | 1 |
| 4 | 4 |
| 3 | 3 |
| 7 | 7 |
| 5 | 5 |
| 9 | 9 |
| 8 | 8 |



Sort each column
in parallel:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 5 | 4 | 4 | 4 | 4 |
| 6 | 5 | 5 | 5 | 6 | 5 | 5 | 5 |
| 7 | 8 | 7 | 7 | 6 | 6 | 7 | 7 |
| 9 | 8 | 8 | 8 | 9 | 9 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

Sort each row in p
$\leftarrow$ or $\rightarrow$ as desire

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 |
| 5 | 5 | 5 | 5 | 5 | 5 |
| 6 | 6 | 7 | 7 | 7 | 7 |
| 8 | 8 | 8 | 8 | 8 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 |

Sort each column
in parallel:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 5 | 4 | 4 | 4 | 4 |
| 6 | 5 | 5 | 5 | 6 | 5 | 5 | 5 |
| 7 | 8 | 7 | 7 | 6 | 6 | 7 | 7 |
| 9 | 8 | 8 | 8 | 9 | 9 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |



Sort each row in parallel,
$\leftarrow$ or $\rightarrow$ as desired:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 5 |
| 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 |
| 6 | 6 | 7 | 7 | 7 | 7 | 7 | 8 |
| 8 | 8 | 8 | 8 | 8 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

Sort each column in parallel:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 5 | 4 | 4 | 4 | 4 |
| 6 | 5 | 5 | 5 | 6 | 5 | 5 | 5 |
| 7 | 8 | 7 | 7 | 6 | 6 | 7 | 7 |
| 9 | 8 | 8 | 8 | 9 | 9 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |



Sort each row in parallel, $\leftarrow$ or $\rightarrow$ as desired:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 5 |
| 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 |
| 6 | 6 | 7 | 7 | 7 | 7 | 7 | 8 |
| 8 | 8 | 8 | 8 | 8 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

h column
el:

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 5 | 4 | 4 | 4 | 4 |
| 5 | 5 | 6 | 5 | 5 | 5 |
| 7 | 7 | 6 | 6 | 7 | 7 |
| 8 | 8 | 9 | 9 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 |



Sort each row in parallel,
$\leftarrow$ or $\rightarrow$ as desired:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 5 |
| 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 |
| 6 | 6 | 7 | 7 | 7 | 7 | 7 | 8 |
| 8 | 8 | 8 | 8 | 8 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

Chips ar
towards
parallelis

GPUs: p
Old Xeo
New Xe

| | |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 3 | 3 |
| 4 | 4 |
| 5 | 5 |
| 7 | 7 |
| 8 | 8 |
| 9 | 9 |



Sort each row in parallel,
$\leftarrow$ or $\rightarrow$ as desired:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 5 |
| 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 |
| 6 | 6 | 7 | 7 | 7 | 7 | 7 | 8 |
| 8 | 8 | 8 | 8 | 8 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

Chips are in fact e
towards having thi
parallelism and co

GPUs: parallel +
Old Xeon Phi: par
New Xeon Phi: pa

Sort each row in parallel,
$\leftarrow$ or $\rightarrow$ as desired:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 5 |
| 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 |
| 6 | 6 | 7 | 7 | 7 | 7 | 7 | 8 |
| 8 | 8 | 8 | 8 | 8 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |



Chips are in fact evolving
towards having this much
parallelism and communicati

GPUs: parallel + global RA
Old Xeon Phi: parallel + rir
New Xeon Phi: parallel + m

Sort each row in parallel,
$\leftarrow$ or $\rightarrow$ as desired:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 5 |
| 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 |
| 6 | 6 | 7 | 7 | 7 | 7 | 7 | 8 |
| 8 | 8 | 8 | 8 | 8 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |



Chips are in fact evolving towards having this much parallelism and communication.

GPUs: parallel + global RAM.
Old Xeon Phi: parallel + ring.
New Xeon Phi: parallel + mesh.

Sort each row in parallel,
$\leftarrow$ or $\rightarrow$ as desired:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 5 |
| 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 |
| 6 | 6 | 7 | 7 | 7 | 7 | 7 | 8 |
| 8 | 8 | 8 | 8 | 8 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |



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Algorithm designers don't even get the right exponent without taking this into account.

Sort each row in parallel,
$\leftarrow$ or $\rightarrow$ as desired:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 4 | 4 | 4 | 4 | 5 |
| 5 | 5 | 5 | 5 | 5 | 5 | 6 | 6 |
| 6 | 6 | 7 | 7 | 7 | 7 | 7 | 8 |
| 8 | 8 | 8 | 8 | 8 | 9 | 9 | 9 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |



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Shock waves from subroutines into high-level algorithm design.