High-speed cryptography, part 1: elliptic-curve formulas

Daniel J. Bernstein
University of Illinois at Chicago &
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Crypto performance problems often lead users to reduce cryptographic security levels or give up on cryptography.

Example 1 (according to Firefox on Linux, 2013.06.24): Google SSL uses RSA-1024.

Security note:
Analyses in 2003 concluded that RSA-1024 was breakable; e.g., 2003 Shamir–Tromer estimated 1 year, $\approx 10^7$ USD.
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Move to RSA-2048 by 2010.

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Example: Curve25519 ECDI
460200 Cortex A8 cycles;
332304 Snapdragon S4 cycle
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$$\left(\frac{X_1}{Z_1}, \frac{Y_1}{Z_1}\right) + \left(\frac{X}{Z}\right.$$

$$\left(\frac{\frac{X_1}{Z_1}\frac{Y_2}{Z_2} + \frac{Y_1}{Z_1}\frac{X_2}{Z_2}}{1 + d\frac{X_1}{Z_1}\frac{X_2}{Z_2}\frac{Y_1}{Z_1}\frac{Y_2}{Z}}\right.$$

$$\frac{\frac{Y_1}{Z_1}\frac{Y_2}{Z_2} - \frac{X_1}{Z_1}\frac{X_2}{Z_2}}{1 - d\frac{X_1}{Z_1}\frac{X_2}{Z_2}\frac{Y_1}{Z_1}\frac{Y_2}{Z}}$$

$$\left(\frac{Z_1Z_2(X_1Y_2 + Y}{Z_1^2Z_2^2 + dX_1X_2}\right.$$

$$\frac{Z_1Z_2(Y_1Y_2 - X}{Z_1^2Z_2^2 - dX_1X_2}$$

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$$\left(\frac{\frac{X_1}{Z_1}\frac{Y_2}{Z_2} + \frac{Y_1}{Z_1}\frac{X_2}{Z_2}}{1 + d\frac{X_1}{Z_1}\frac{X_2}{Z_2}\frac{Y_1}{Z_1}\frac{Y_2}{Z_2}}, \frac{\frac{Y_1}{Z_1}\frac{Y_2}{Z_2} - \frac{X_1}{Z_1}\frac{X_2}{Z_2}}{1 - d\frac{X_1}{Z_1}\frac{X_2}{Z_2}\frac{Y_1}{Z_1}\frac{Y_2}{Z_2}}\right) =$$

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i.e. $\Big(\frac{X_1}{Z_1}$

$= \Big(\frac{X_3}{Z_3},$

where

$F = Z_1^2$

$G = Z_1^2$

$X_3 = Z_1$

$Y_3 = Z_1$

$Z_3 = F$

Input to

$X_1, Y_1,$

Output

$X_3, Y_3,$

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$F = Z_1^2 Z_2^2 - d X_1$

$G = Z_1^2 Z_2^2 + d X_1$

$X_3 = Z_1 Z_2 (X_1 Y_2$

$Y_3 = Z_1 Z_2 (Y_1 Y_2 -$

$Z_3 = FG.$

Input to addition a

$X_1, Y_1, Z_1, X_2, Y_2,$

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Input to addition algorithm:
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$$\ldots 1\Big) + \left(\frac{X_2}{Z_2}, \frac{Y_2}{Z_2}\right) =$$

$$\ldots \frac{\ldots_2 + \frac{Y_1}{Z_1}\frac{X_2}{Z_2}}{\ldots \frac{X_1}{Z_1}\frac{X_2}{Z_2}\frac{Y_1}{Z_1}\frac{Y_2}{Z_2}},$$

$$\ldots \frac{\ldots_2 - \frac{X_1}{Z_1}\frac{X_2}{Z_2}}{\ldots \frac{X_1}{Z_1}\frac{X_2}{Z_2}\frac{Y_1}{Z_1}\frac{Y_2}{Z_2}}\Bigg) =$$

$$\ldots \frac{(X_1Y_2 + Y_1X_2)}{\ldots + dX_1X_2Y_1Y_2},$$

$$\ldots \frac{(Y_1Y_2 - X_1X_2)}{\ldots - dX_1X_2Y_1Y_2}\Bigg)$$

i.e. $\left(\frac{X_1}{Z_1}, \frac{Y_1}{Z_1}\right) + \left(\frac{X_2}{Z_2}, \frac{Y_2}{Z_2}\right)$

$= \left(\frac{X_3}{Z_3}, \frac{Y_3}{Z_3}\right)$

where

$F = Z_1^2 Z_2^2 - dX_1X_2Y_1Y_2,$
$G = Z_1^2 Z_2^2 + dX_1X_2Y_1Y_2,$
$X_3 = Z_1Z_2(X_1Y_2 + Y_1X_2)F,$
$Y_3 = Z_1Z_2(Y_1Y_2 - X_1X_2)G,$
$Z_3 = FG.$

Input to addition algorithm:
$X_1, Y_1, Z_1, X_2, Y_2, Z_2$.
Output from addition algorithm:
X_3, Y_3, Z_3. No divisions needed!

Save mu

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$A = Z_1$
$C = X_1$
$D = Y_1 \cdot$
$E = d \cdot$
$F = B -$
$X_3 = A$
$Y_3 = A \cdot$
$Z_3 = F$

Cost: 11

Can do

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s input:

$$\frac{\ }{\ 2}, \frac{Y_2}{Z_2}\right) =$$

$$\frac{\ }{\ 2},$$

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$$\frac{{}_1X_2)}{Y_1Y_2},$$

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Save multiplicatio
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$X_3 = A \cdot F \cdot (X_1 \cdot$
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Cost: $11\mathbf{M} + 1\mathbf{S} +$
Can do better: 10

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$$\left(\frac{X_1}{Z_1}, \frac{Y_1}{Z_1}\right) + \left(\frac{X_2}{Z_2}, \frac{Y_2}{Z_2}\right)$$

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Can do better: $10\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$.

Faster d

(x_1, y_1)
$((x_1y_1+$
$(y_1y_1-$
$((2x_1y_1)$
$(y_1^2-x_1^2$

$x_1^2 + y_1^2$
(x_1, y_1)
$((2x_1y_1)$
$(y_1^2-x_1^2$

Again el
using $\mathbf{P}^2$
Much fa
Useful:

$\left(\frac{X_2}{Z_2}, \frac{Y_2}{Z_2}\right)$

$X_2 Y_1 Y_2,$

$X_2 Y_1 Y_2,$

$+ Y_1 X_2) F,$

$- X_1 X_2) G,$

algorithm:

$Z_2.$

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visions needed!

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Cost: $11\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$.

Can do better: $10\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$.

<u>Faster doubling</u>

$(x_1, y_1) + (x_1, y_1)$

$((x_1 y_1 + y_1 x_1)/(1$

$(y_1 y_1 - x_1 x_1)/(1$

$((2 x_1 y_1)/(1 + d x_1^2$

$(y_1^2 - x_1^2)/(1 - d x$

$x_1^2 + y_1^2 = 1 + d x_1^2$

$(x_1, y_1) + (x_1, y_1)$

$((2 x_1 y_1)/(x_1^2 + y_1^2$

$(y_1^2 - x_1^2)/(2 - x_1^2$

Again eliminate di

using $\mathbf{P}^2$: only $3\mathbf{M}$

Much faster than

Useful: many doul

thm:

ded!

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Can do better: $10\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$.

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$(x_1, y_1) + (x_1, y_1) =$
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$(y_1y_1 - x_1x_1)/(1 - dx_1x_1y_1$
$((2x_1y_1)/(1 + dx_1^2y_1^2),$
$(y_1^2 - x_1^2)/(1 - dx_1^2y_1^2)).$

$x_1^2 + y_1^2 = 1 + dx_1^2y_1^2$ so
$(x_1, y_1) + (x_1, y_1) =$
$((2x_1y_1)/(x_1^2 + y_1^2),$
$(y_1^2 - x_1^2)/(2 - x_1^2 - y_1^2)).$

Again eliminate divisions using $\mathbf{P}^2$: only $3\mathbf{M} + 4\mathbf{S}$.
Much faster than addition.
Useful: many doublings in E

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$(x_1, y_1) + (x_1, y_1) =$
$((x_1 y_1 + y_1 x_1)/(1 + d x_1 x_1 y_1 y_1),$
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ing common
essions:

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$- E$; $G = B + E$;

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$G \cdot (D - C)$;

$\cdot G$.

$1\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$.

better: $10\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$.

Faster doubling

$$(x_1, y_1) + (x_1, y_1) = \\ ((x_1y_1 + y_1x_1)/(1 + dx_1x_1y_1y_1), \\ (y_1y_1 - x_1x_1)/(1 - dx_1x_1y_1y_1)) = \\ ((2x_1y_1)/(1 + dx_1^2y_1^2), \\ (y_1^2 - x_1^2)/(1 - dx_1^2y_1^2)).$$

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More ad

Dual ad

(x_1, y_1)

$((x_1y_1 +$

$(x_1y_1 -$

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A^2;

$B + E$;

$Y_2 + Y_1 \cdot X_2$);

C);

$+ 1\mathbf{D}$.

$\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$.

Faster doubling

$(x_1, y_1) + (x_1, y_1) =$
$((x_1y_1{+}y_1x_1)/(1{+}dx_1x_1y_1y_1),$
$(y_1y_1{-}x_1x_1)/(1{-}dx_1x_1y_1y_1)) =$
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More addition stra

Dual addition form
$(x_1, y_1) + (x_2, y_2)$
$((x_1y_1 + x_2y_2)/(x$
$(x_1y_1 - x_2y_2)/(x$
Low degree, no ne

Warning: fails for
Is this really "addi
Most EC formulas

X_2);

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More addition strategies

Dual addition formula:
$(x_1, y_1) + (x_2, y_2) =$
$((x_1y_1 + x_2y_2)/(x_1x_2 + y_1y$
$(x_1y_1 - x_2y_2)/(x_1y_2 - x_2y$
Low degree, no need for d.

Warning: fails for doubling!
Is this really "addition"?
Most EC formulas have failu

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More coordinate systems:
Inverted: $x = Z/X$, $y = Z/Y$.
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oubling

$+ (x_1, y_1) =$

$y_1 x_1)/(1{+}dx_1 x_1 y_1 y_1),$

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many doublings in ECC.

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$v^2 = u^3$

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$-dx_1x_1y_1y_1)) =$
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More elliptic curve

Edwards curves ar
Easiest way to unc
elliptic curves is E

Geometrically, all
are Edwards curve

Algebraically,
more elliptic curve

Every odd-char cu
expressed as Weie
$v^2 = u^3 + a_2u^2 +$

Warning: "Weiers
different meaning

$y_1)$,

$y_1)) =$

ECC.

More addition strategies

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More elliptic curves

Edwards curves are elliptic.

Easiest way to understand elliptic curves is Edwards.

Geometrically, all elliptic cur are Edwards curves.

Algebraically, more elliptic curves exist.

Every odd-char curve can be expressed as Weierstrass cur

$v^2 = u^3 + a_2 u^2 + a_4 u + a_6$

Warning: "Weierstrass" has different meaning in char 2.

More addition strategies

Dual addition formula:
$(x_1, y_1) + (x_2, y_2) =$
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Geometrically, all elliptic curves
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Algebraically,
more elliptic curves exist.

Every odd-char curve can be
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$v^2 = u^3 + a_2u^2 + a_4u + a_6$.

Warning: "Weierstrass" has
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$+ (x_2, y_2) =$

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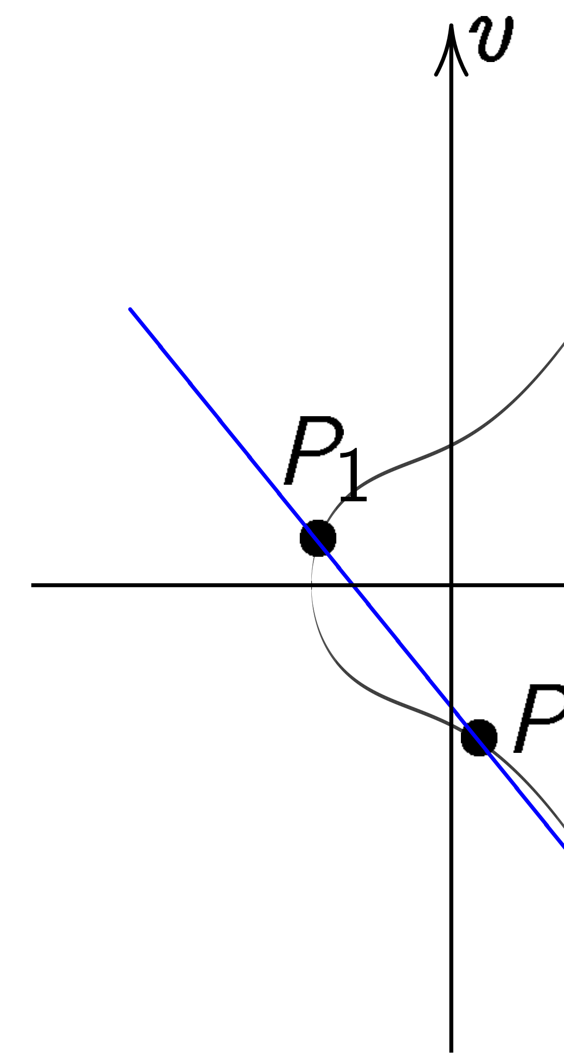

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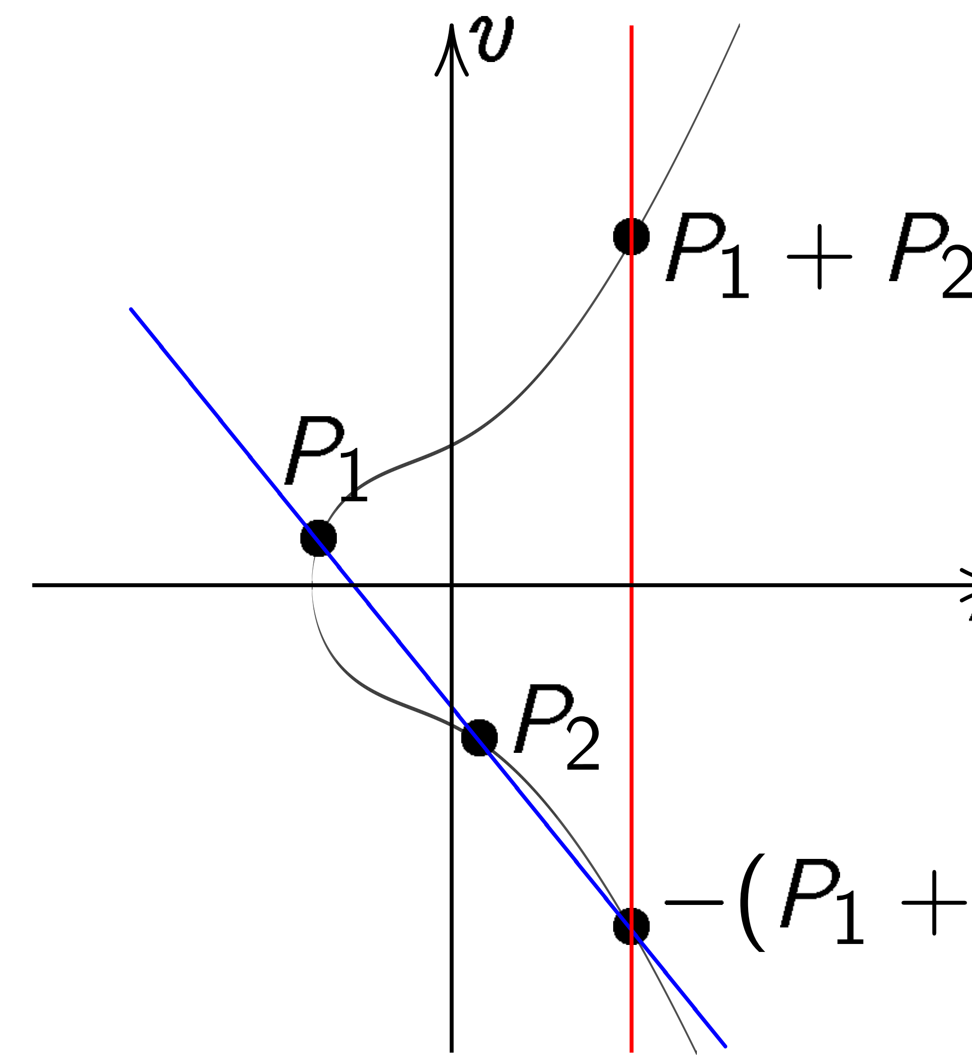

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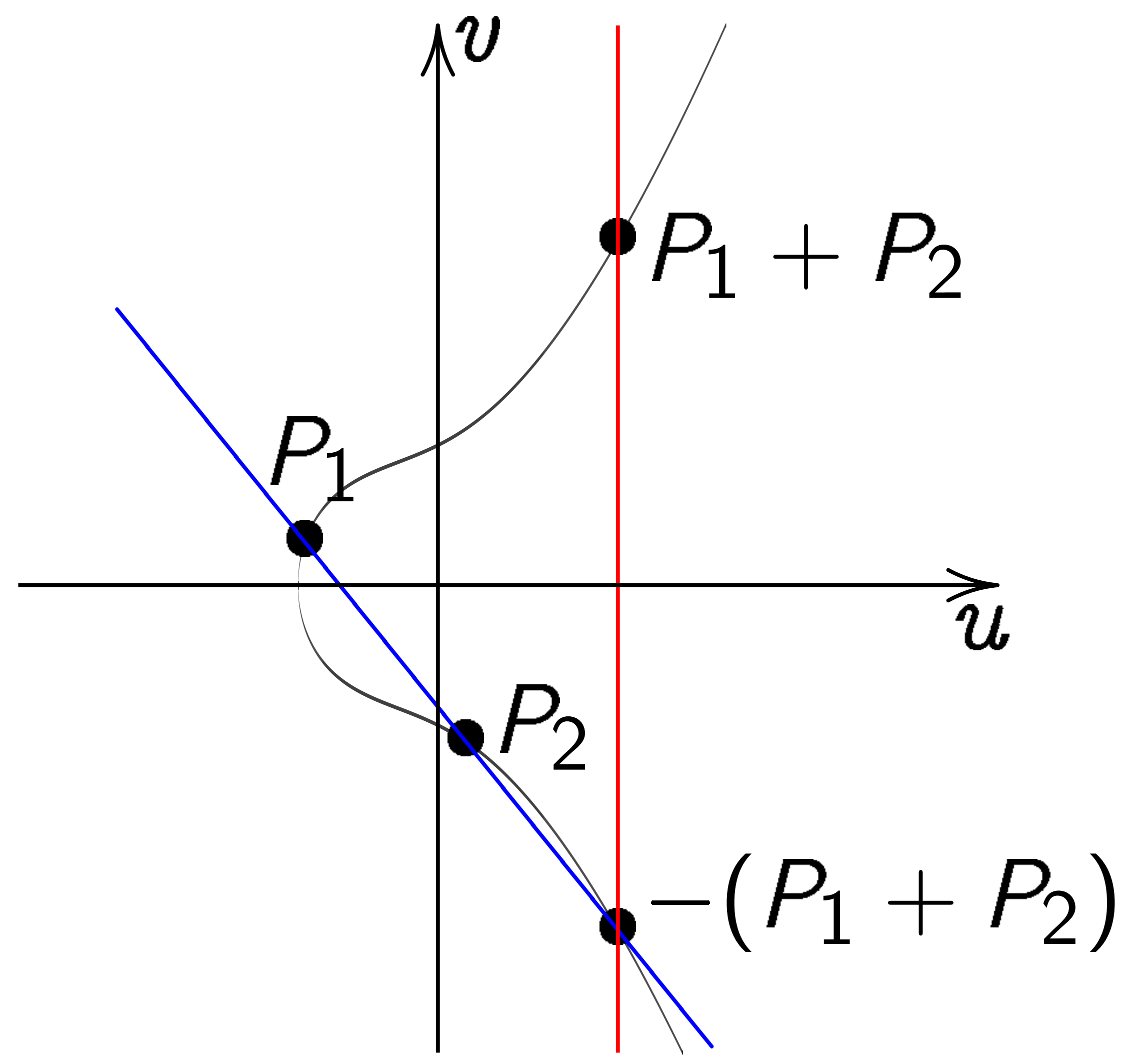

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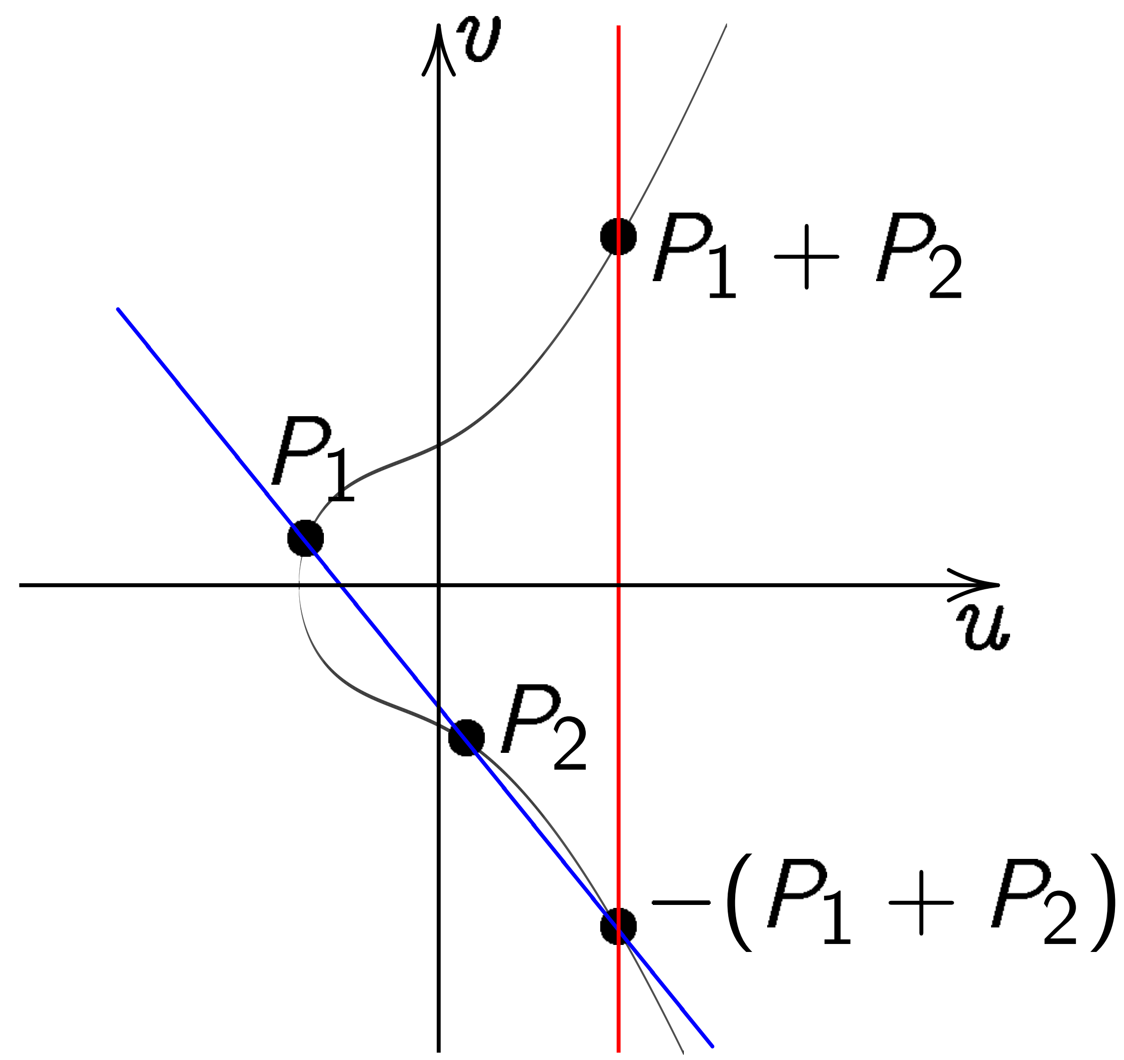

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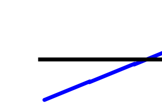

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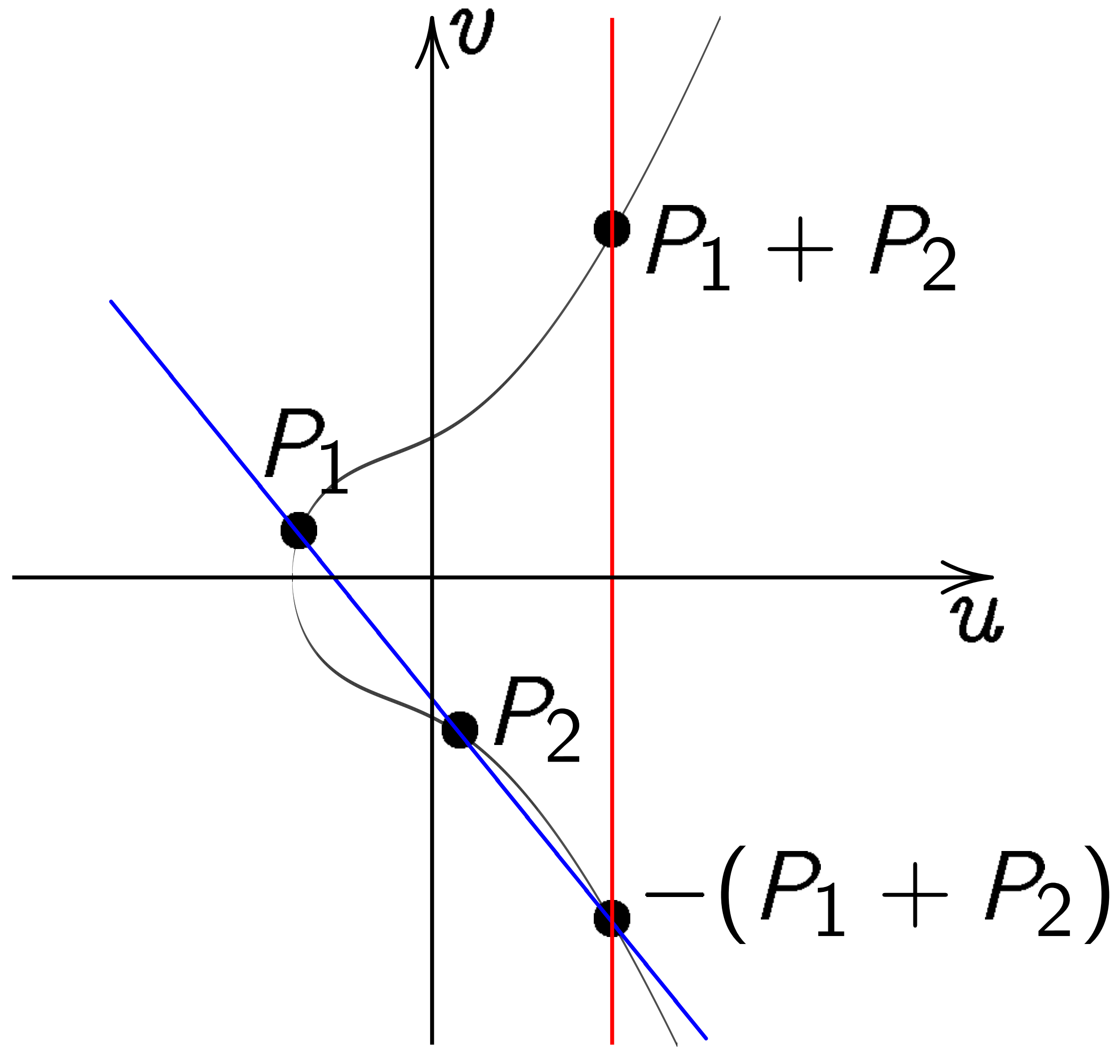

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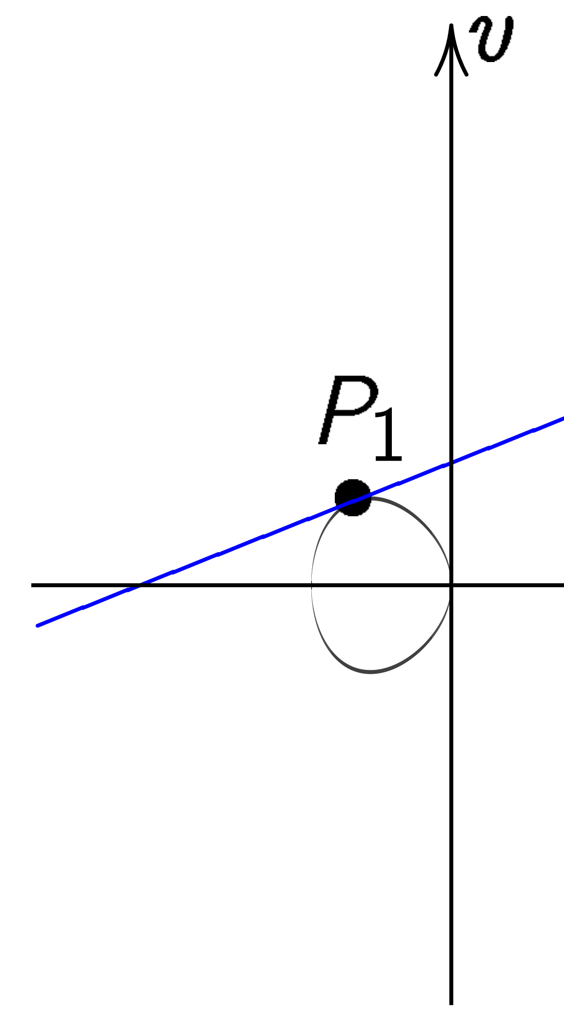

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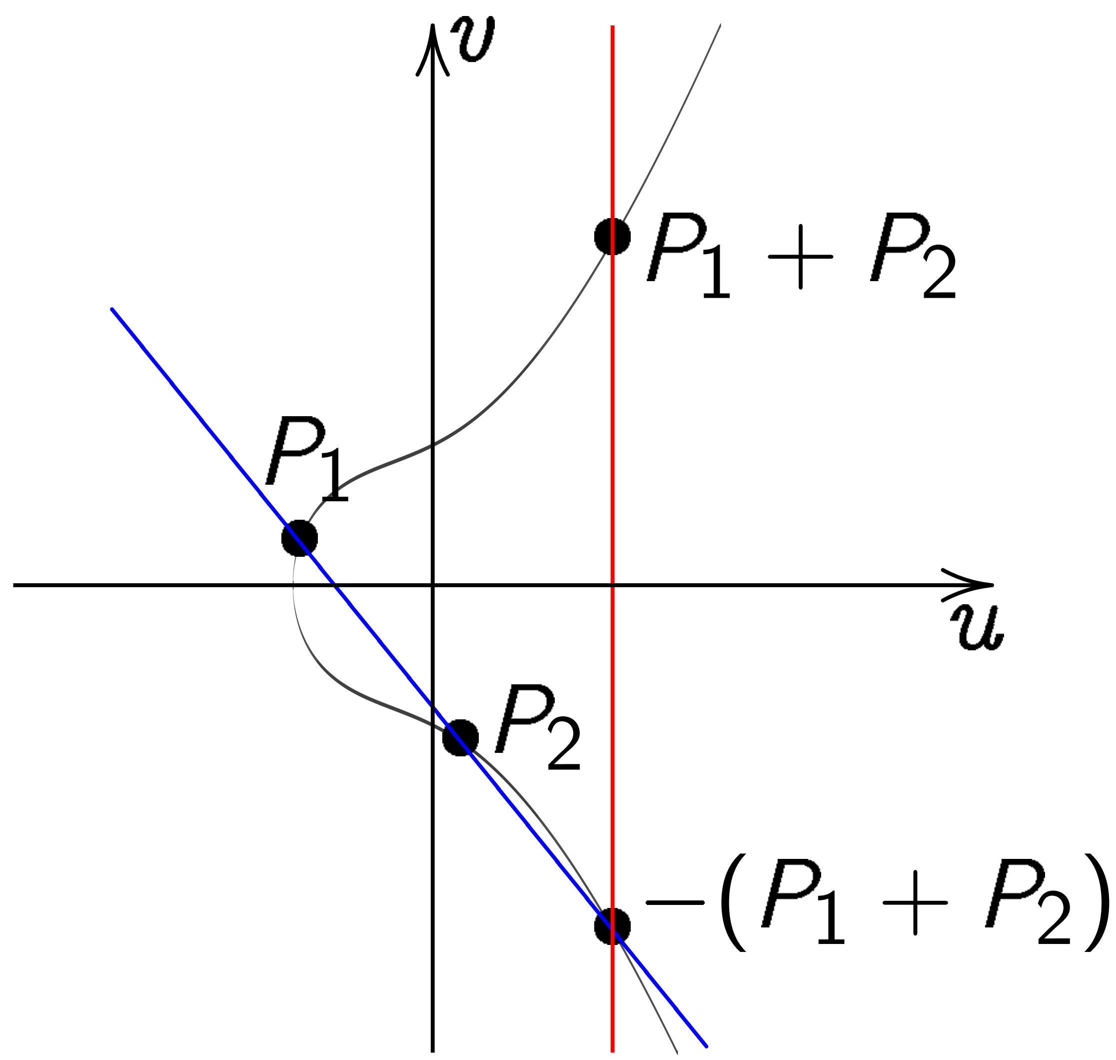

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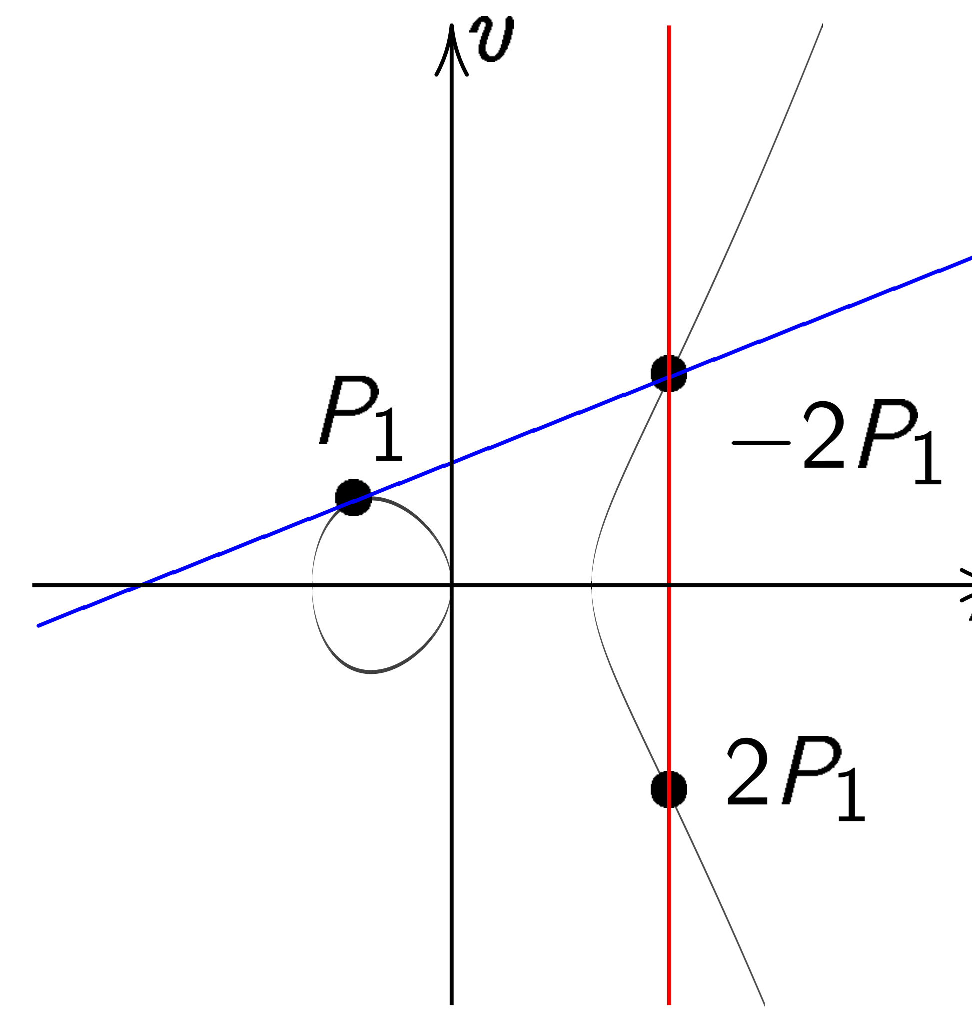

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Addition on Weierstrass curve

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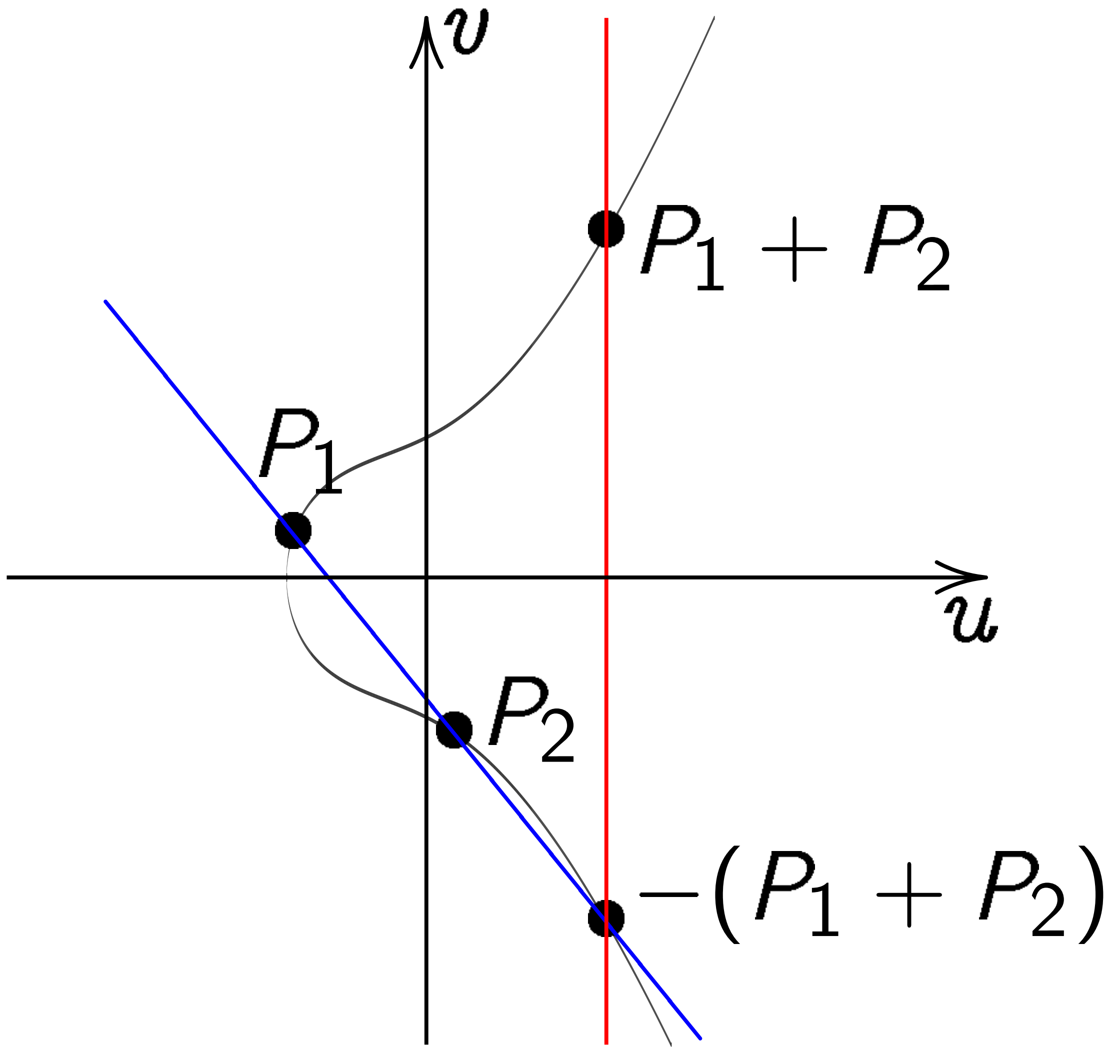

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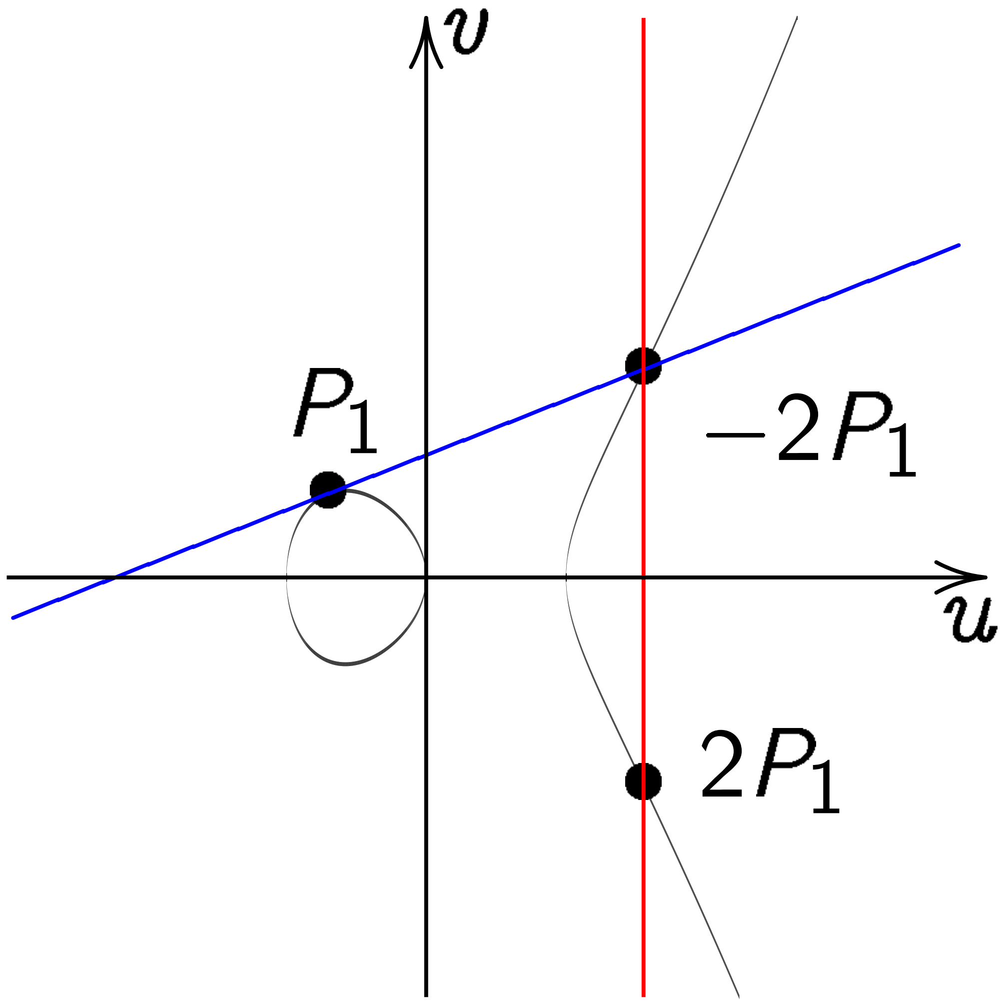

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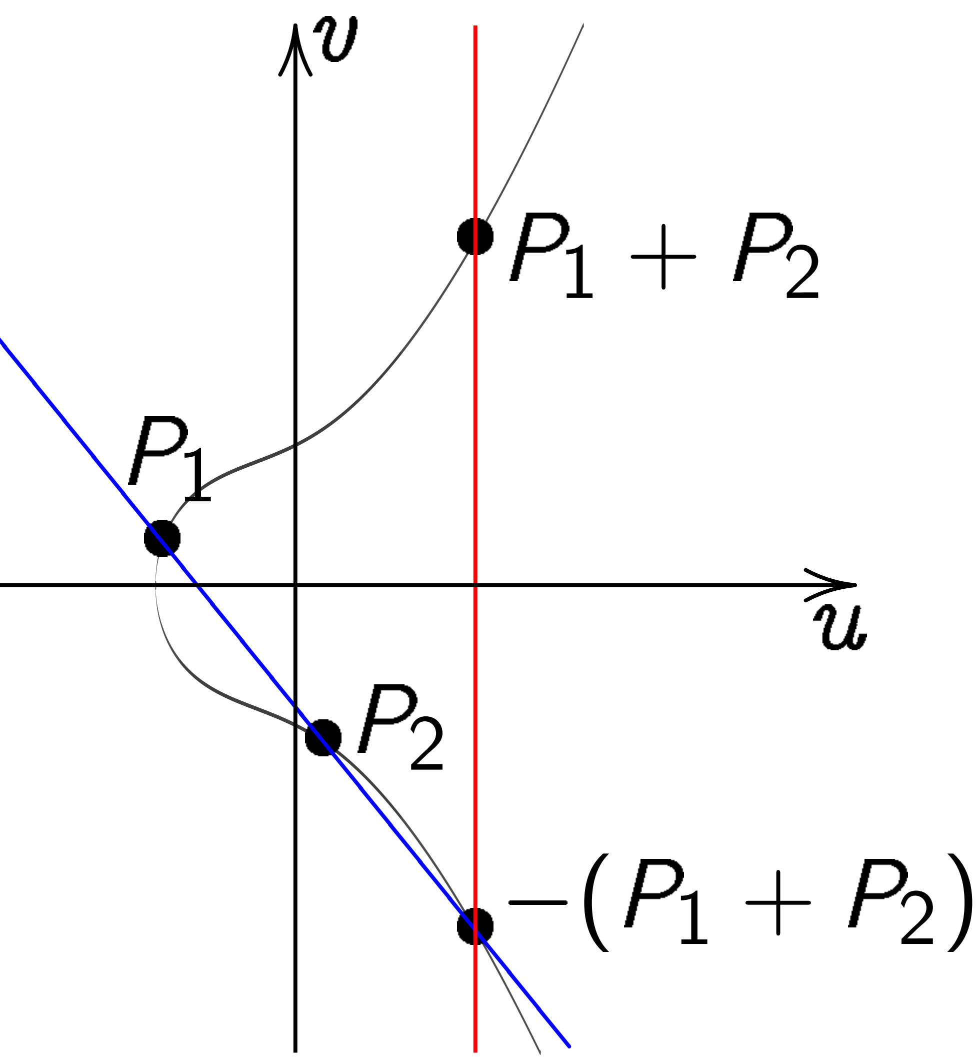

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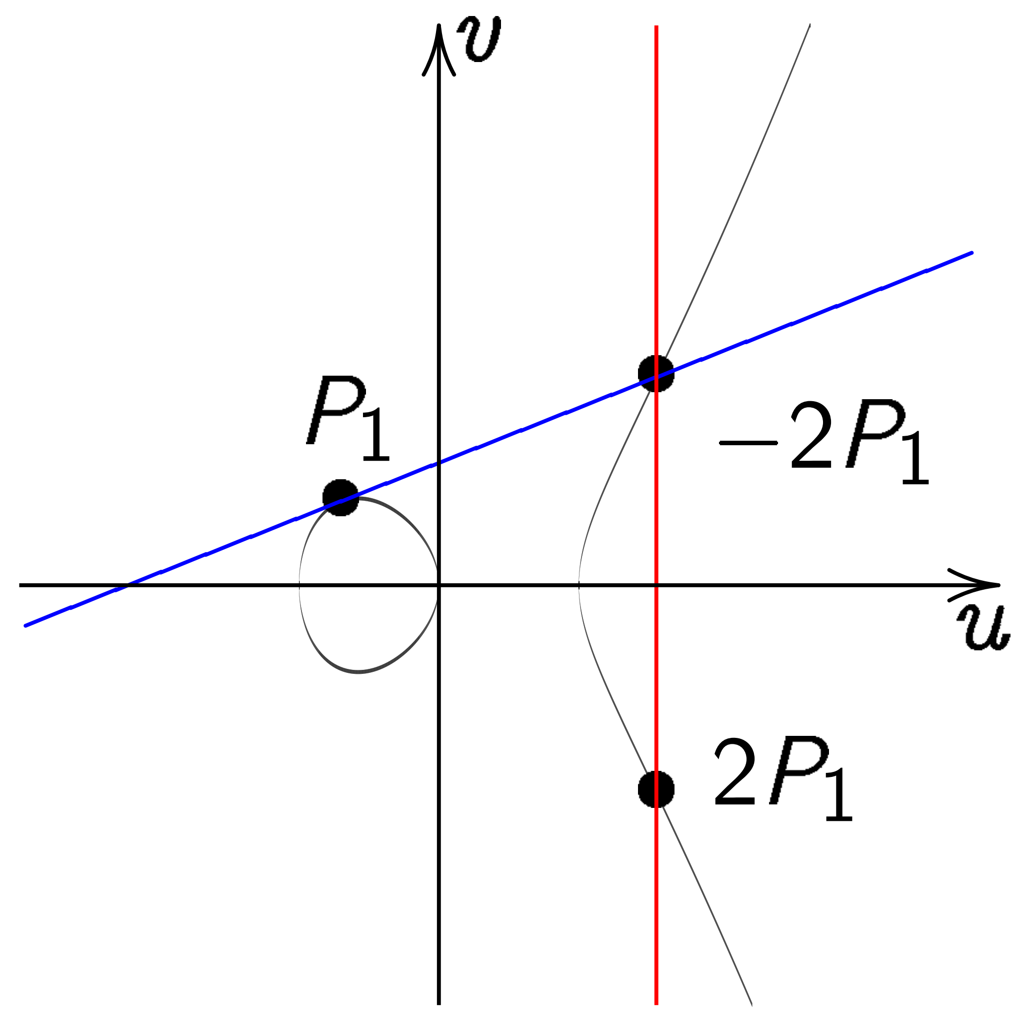

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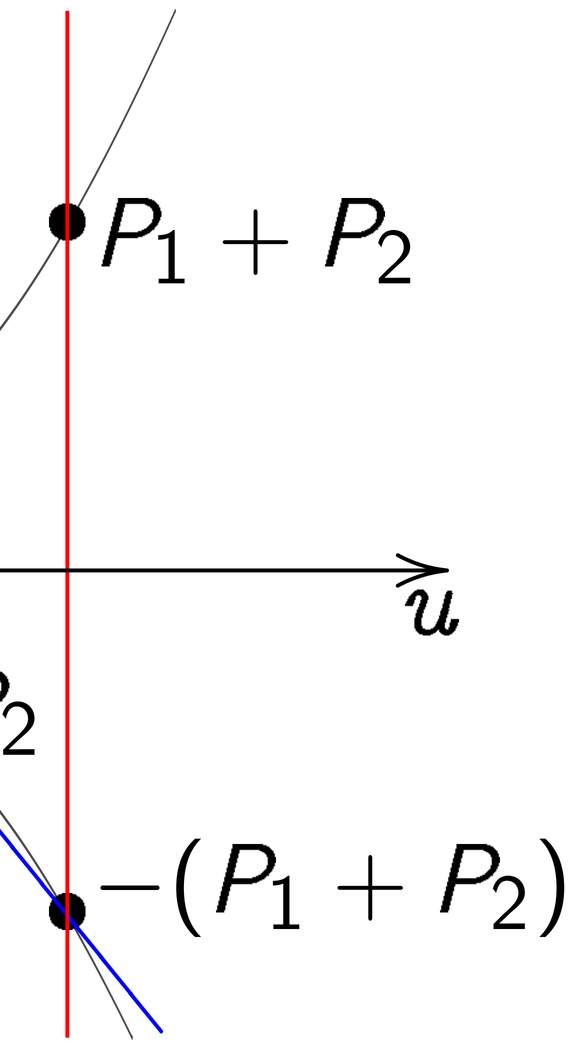

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Doubling on Weierstrass curve

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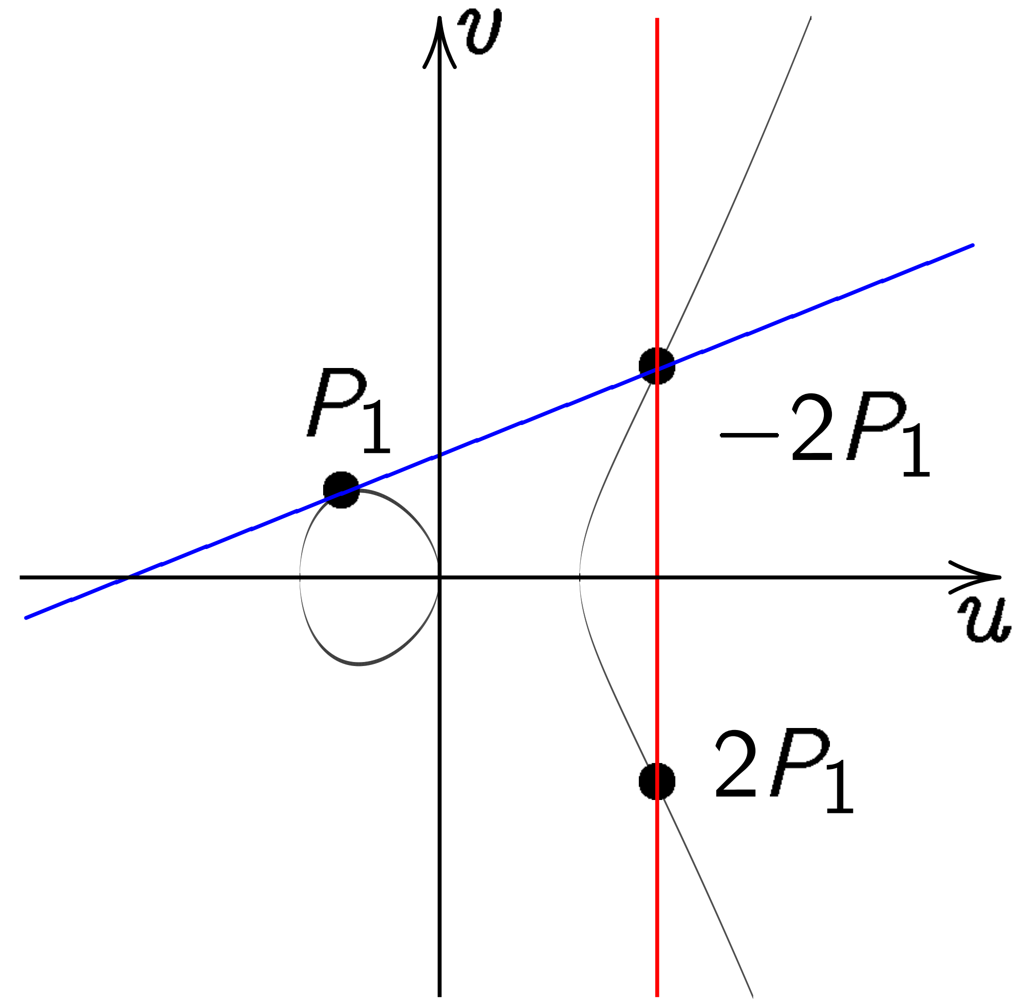

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Doubling on Weierstrass curve

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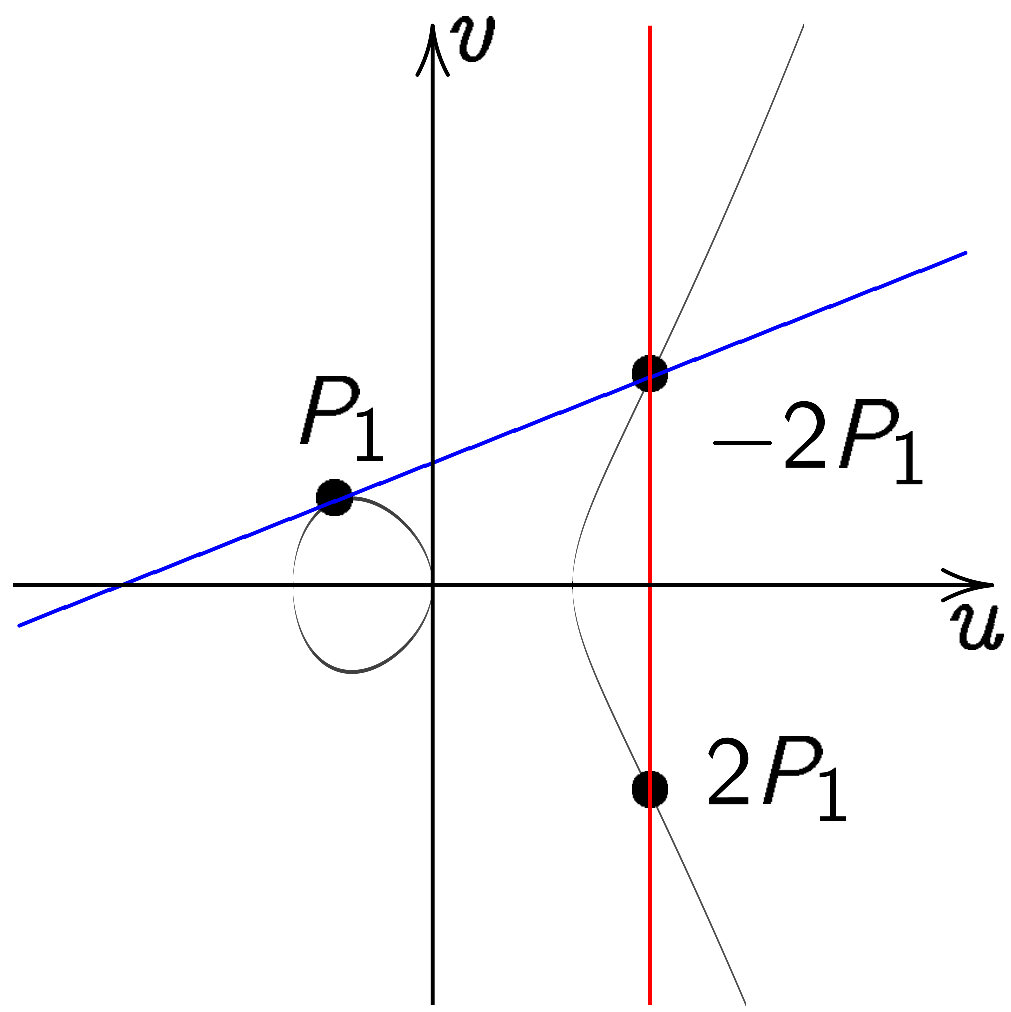

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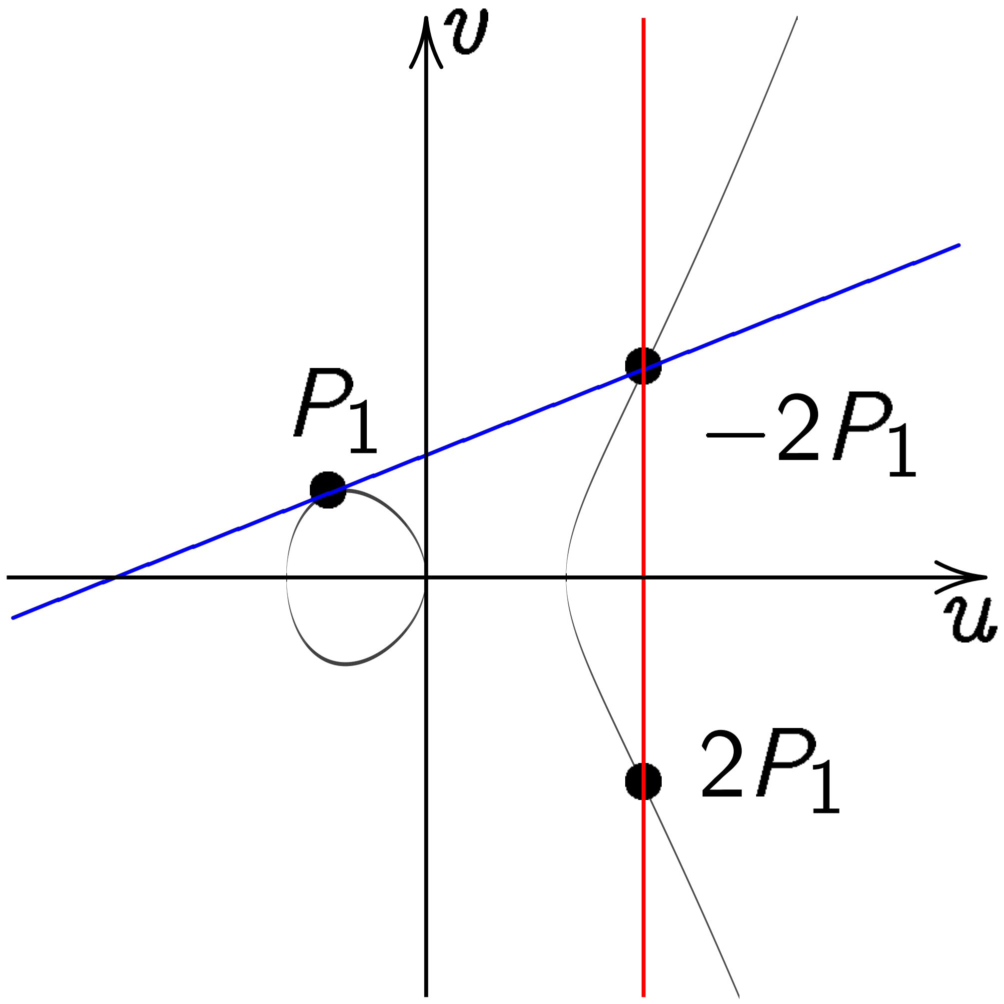

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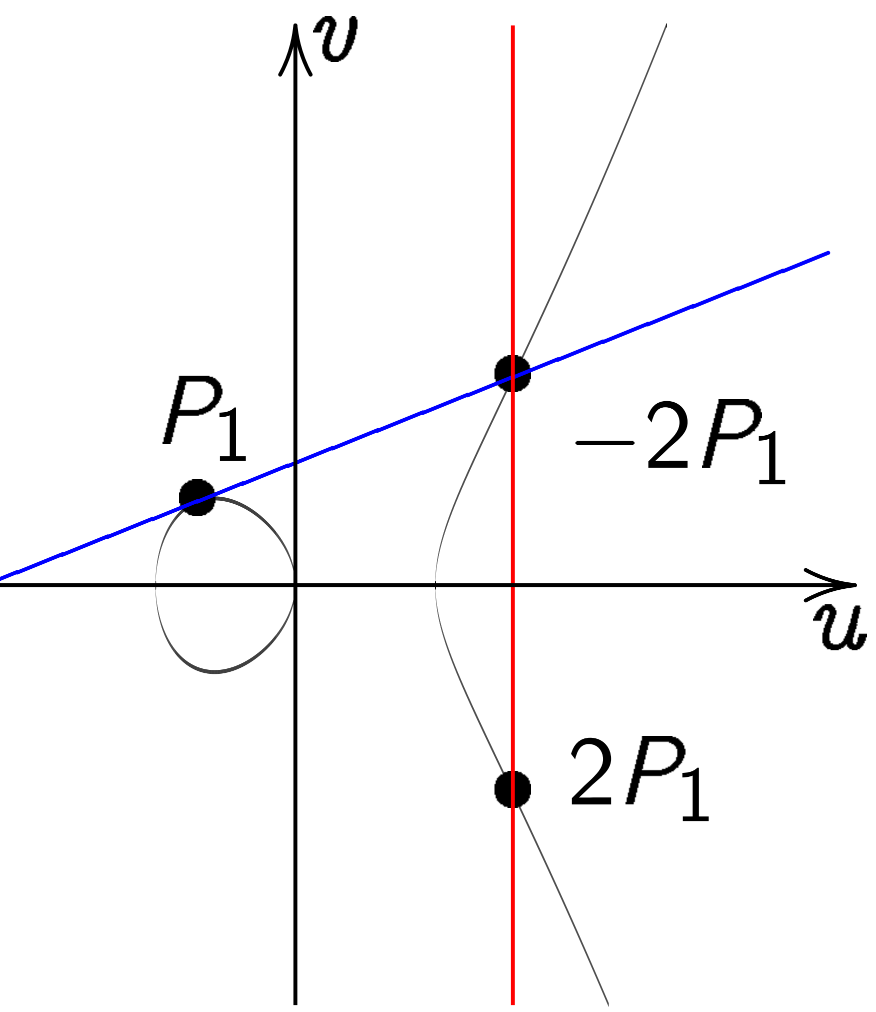

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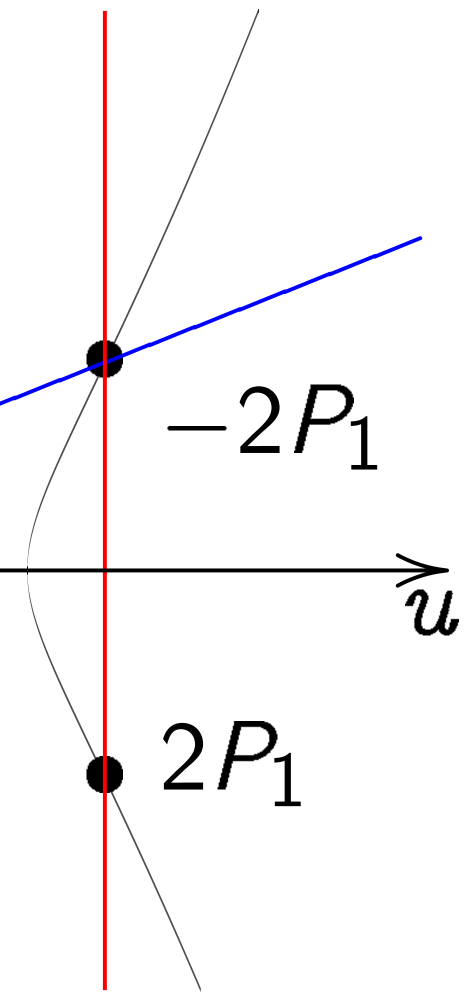

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Starting from poin
on $x^2 + y^2 = 1 +$

Define $A = 2(1 +$
$B = 4/(1 - d)$;
$u = (1 + y)/(B(1$
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$v^2 = u^3 + (A/B)u$

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Maps Edwards to Weierstras
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Easily invert this map:
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$1 - d)$;
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$= (1 + y)/(Bx(1 - y))$.
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<u>Optimizing Jacobian coordin</u>

For “traditional” $(X/Z^2, Y/$
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state explicit formulas using
10**M** for DBL; 16**M** for ADD

Consequence:
$$\approx \left(10 \lg n + 16 \frac{\lg n}{\lg \lg n}\right) \mathbf{M}$$
to compute $n, P \mapsto nP$
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Notation: $\lg = \log_2$.

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Optimizing Jacobian coordinates

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Squaring is faster

Here are the DBL

$S = 4X_1 \cdot Y_1^2;$

$M = 3X_1^2 + aZ$

$T = M^2 - 2S;$

$X_3 = T;$

$Y_3 = M \cdot (S - T$

$Z_3 = 2Y_1 \cdot Z_1.$

Total cost 3**M** + 6

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Squaring is faster than **M**.

Here are the DBL formulas:

$S = 4X_1 \cdot Y_1^2;$
$M = 3X_1^2 + aZ_1^4;$
$T = M^2 - 2S;$
$X_3 = T;$
$Y_3 = M \cdot (S - T) - 8Y_1^4;$
$Z_3 = 2Y_1 \cdot Z_1.$

Total cost $3\mathbf{M} + 6\mathbf{S} + 1\mathbf{D}$ w

S is the cost of squaring in

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The squarings produce $X_1^2, Y_1^2, Y_1^4, Z_1^2, Z_1^4, M^2$.

Optimizing Jacobian coordinates

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$$Y_3 = M \cdot (S - T) - 8Y_1^4;$$
$$Z_3 = 2Y_1 \cdot Z_1.$$

Total cost $3\mathbf{M} + 6\mathbf{S} + 1\mathbf{D}$ where **S** is the cost of squaring in $\mathbf{F}_q$, **D** is the cost of multiplying by a.

The squarings produce $X_1^2, Y_1^2, Y_1^4, Z_1^2, Z_1^4, M^2$.

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Curve-choice advice from 1986 Chudnovsky–Chudnovs

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But "it is even smarter" to choose curve with $a = -$

If $a = -3$ then $M = 3(X_1^2 -$
$= 3(X_1 - Z_1^2) \cdot (X_1 + Z_1^2)$.
Replace $2\mathbf{S}$ with $1\mathbf{M}$.

Now DBL costs $4\mathbf{M} + 4\mathbf{S}$.

Squaring is faster than **M**.

Here are the DBL formulas:

$$S = 4X_1 \cdot Y_1^2;$$
$$M = 3X_1^2 + aZ_1^4;$$
$$T = M^2 - 2S;$$
$$X_3 = T;$$
$$Y_3 = M \cdot (S - T) - 8Y_1^4;$$
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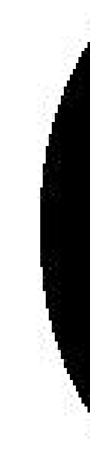

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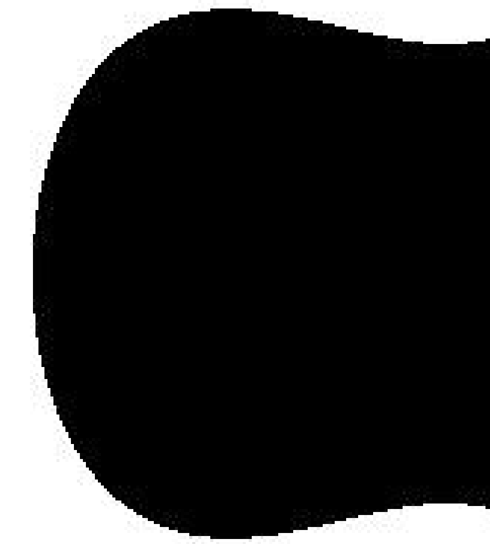

$y^2 = x^3 - 0.4x +$

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Compare to speeds for Edwards curves $x^2 + y^2 = 1 + dx^2y^2$ in projective coordinates (2007 Bernstein–Lange):

$3\mathbf{M} + 4\mathbf{S}$ for DBL.

$10\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$ for ADD.

$9\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$ for mADD.

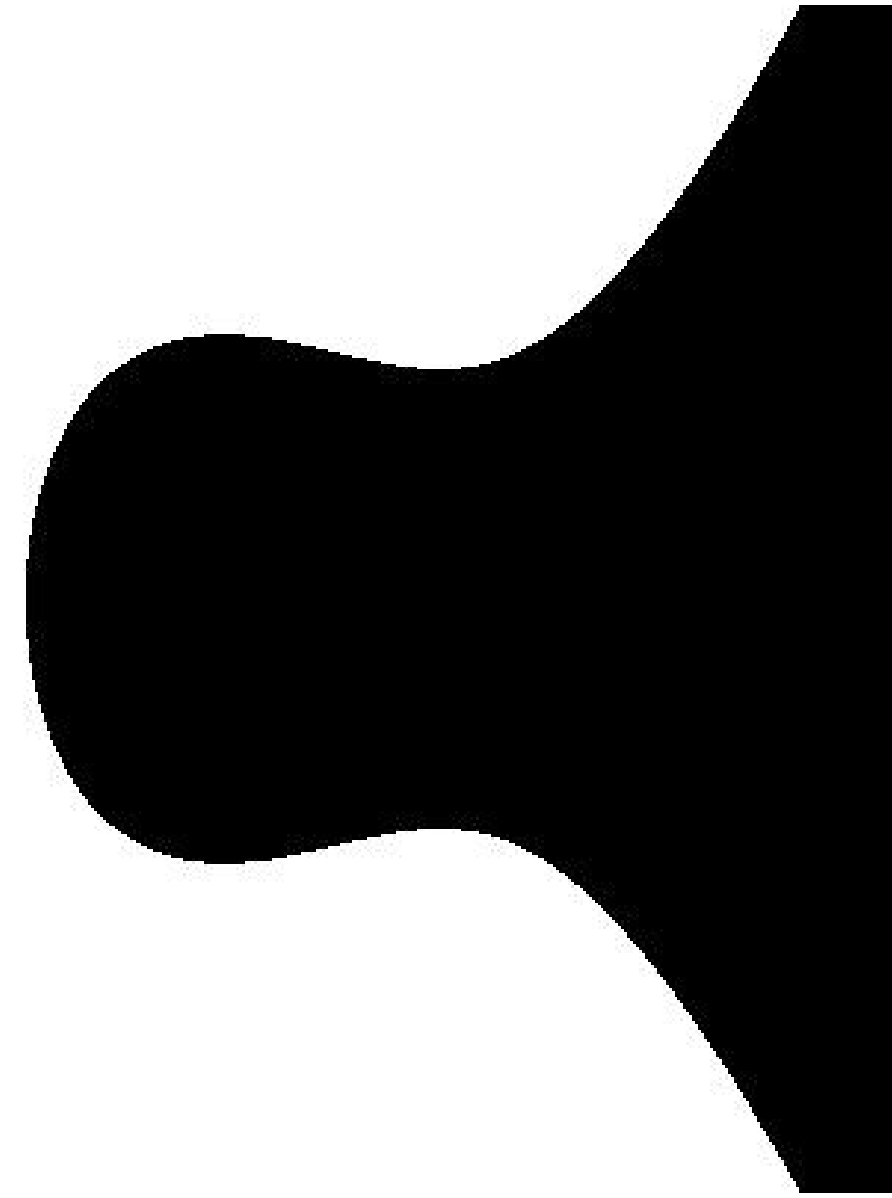
Inverted Edwards coordinates (2007 Bernstein–Lange):

$3\mathbf{M} + 4\mathbf{S} + 1\mathbf{D}$ for DBL.

$9\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$ for ADD.

$8\mathbf{M} + 1\mathbf{S} + 1\mathbf{D}$ for mADD.

Even better speeds from extended/completed coordinates (2008 Hisil–Wong–Carter–Dawson).



$y^2 = x^3 - 0.4x + 0.7$

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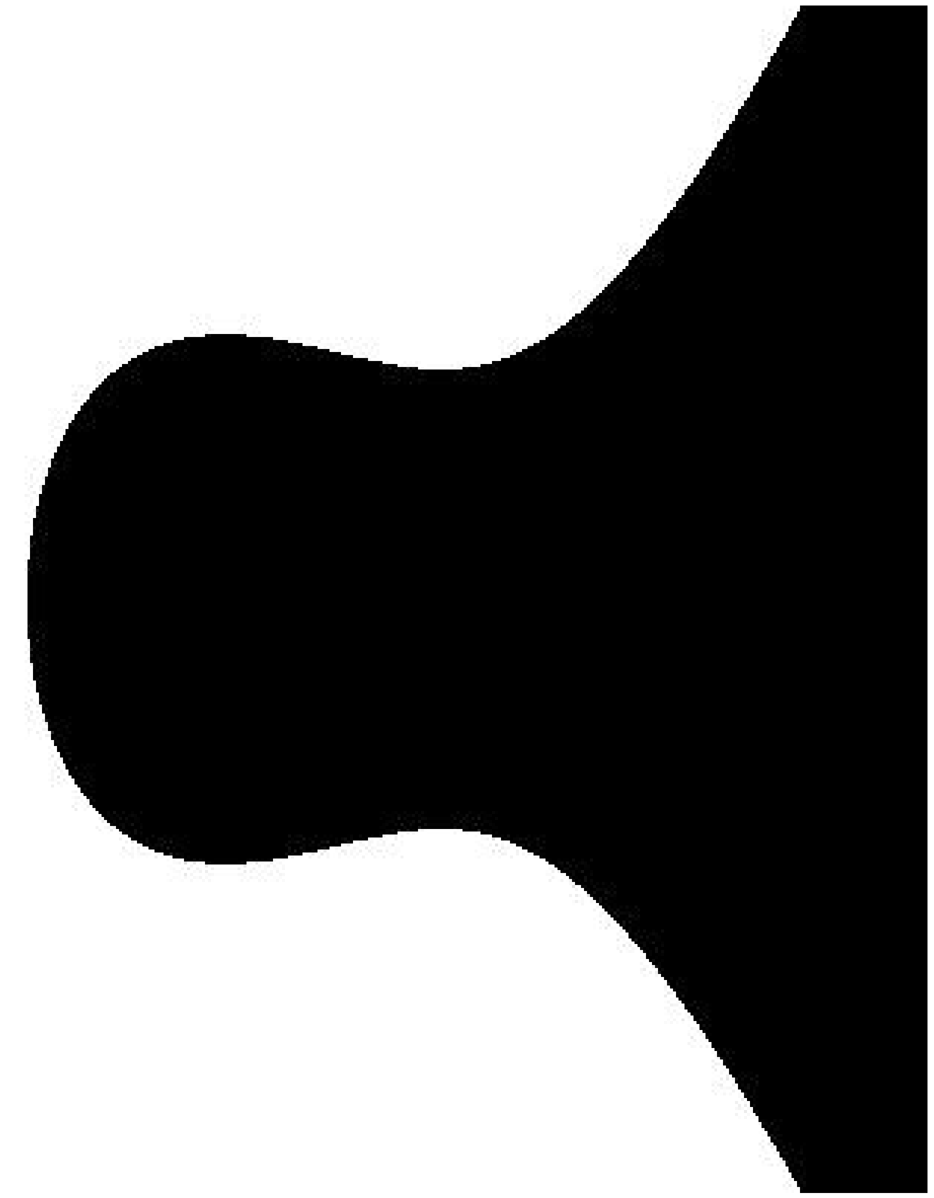

$y^2 = x^3 - 0.4x + 0.7$

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$^2 + y^2 = 1 + dx^2y^2$

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S for DBL.

1**S** + 1**D** for ADD.

S + 1**D** for mADD.

Edwards coordinates

ernstein–Lange):

S + 1**D** for DBL.

S + 1**D** for ADD.

S + 1**D** for mADD.

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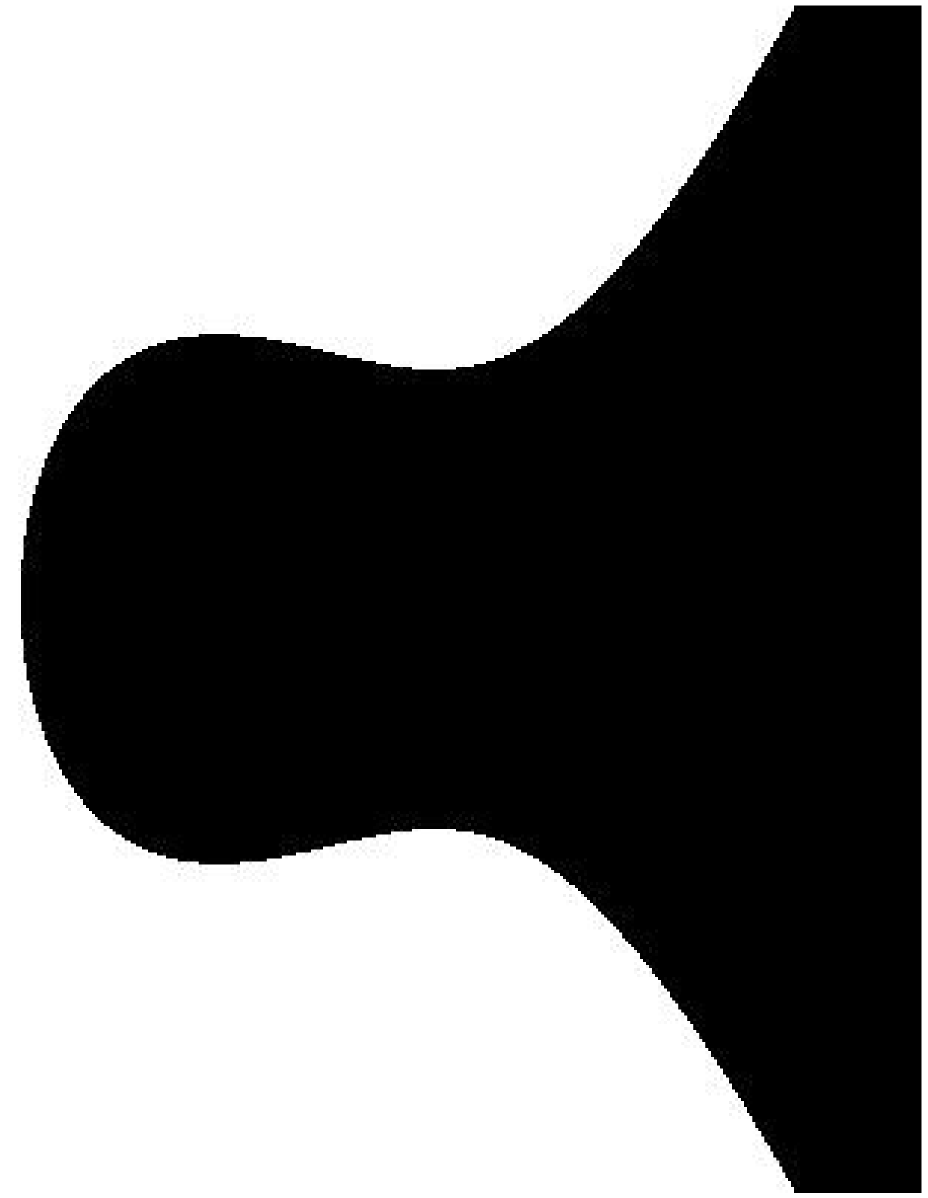

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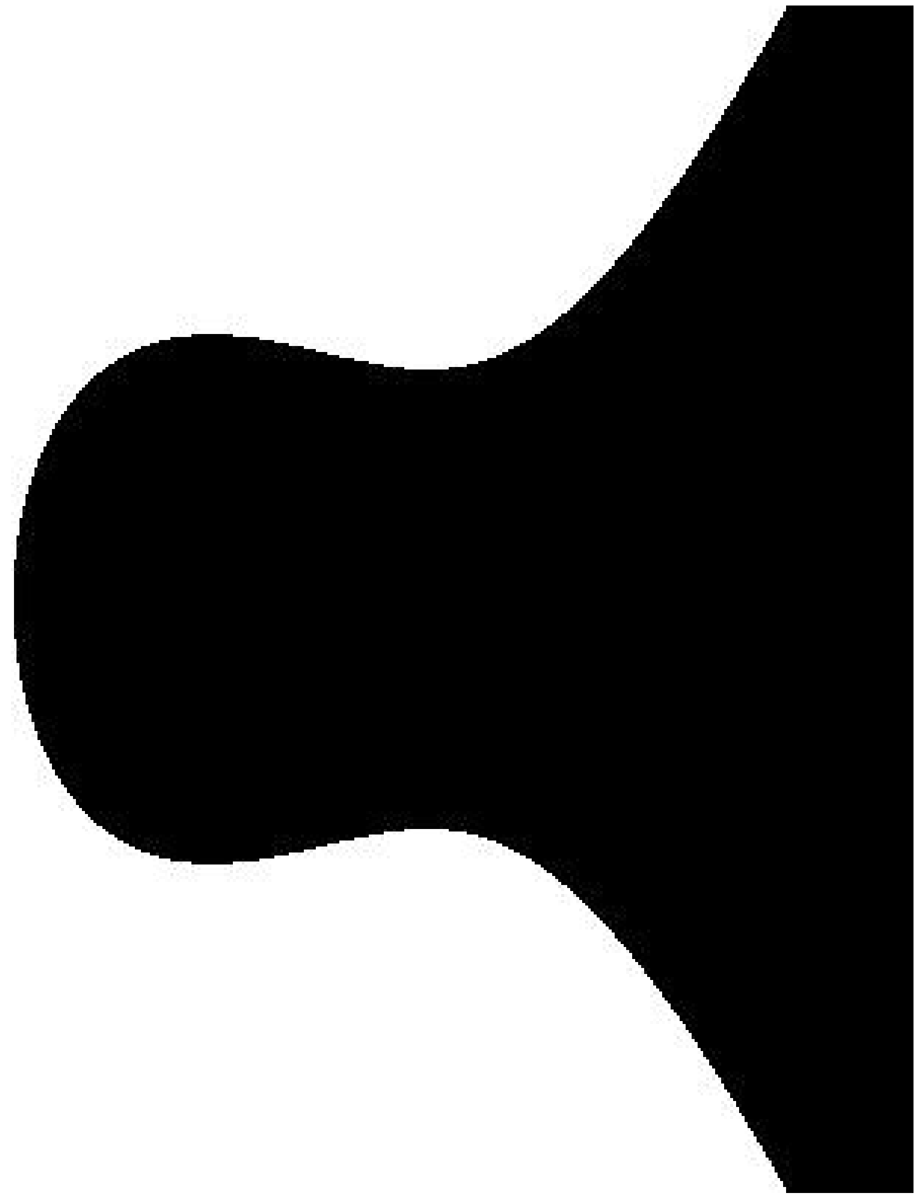

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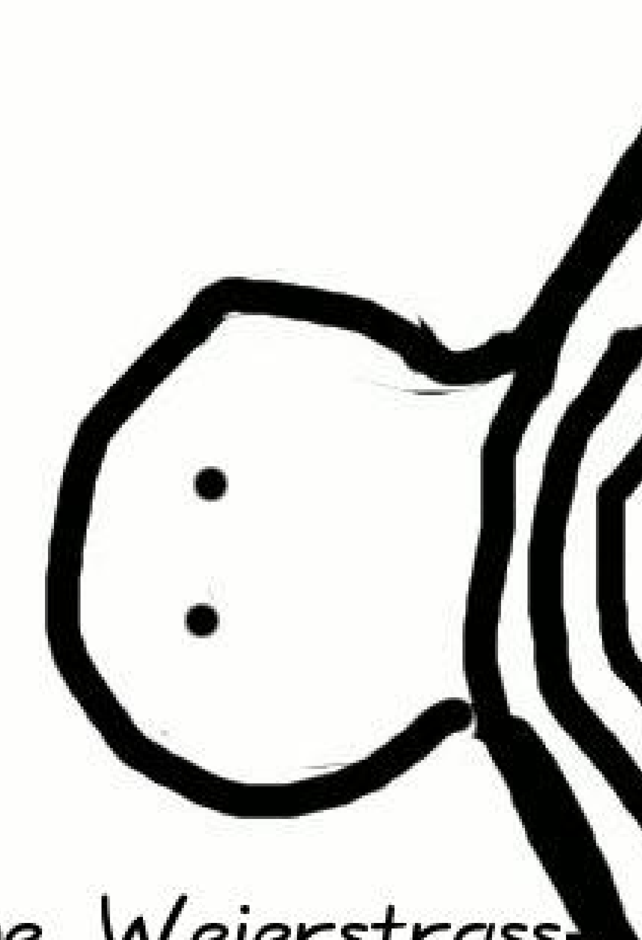

The Weierstrass-
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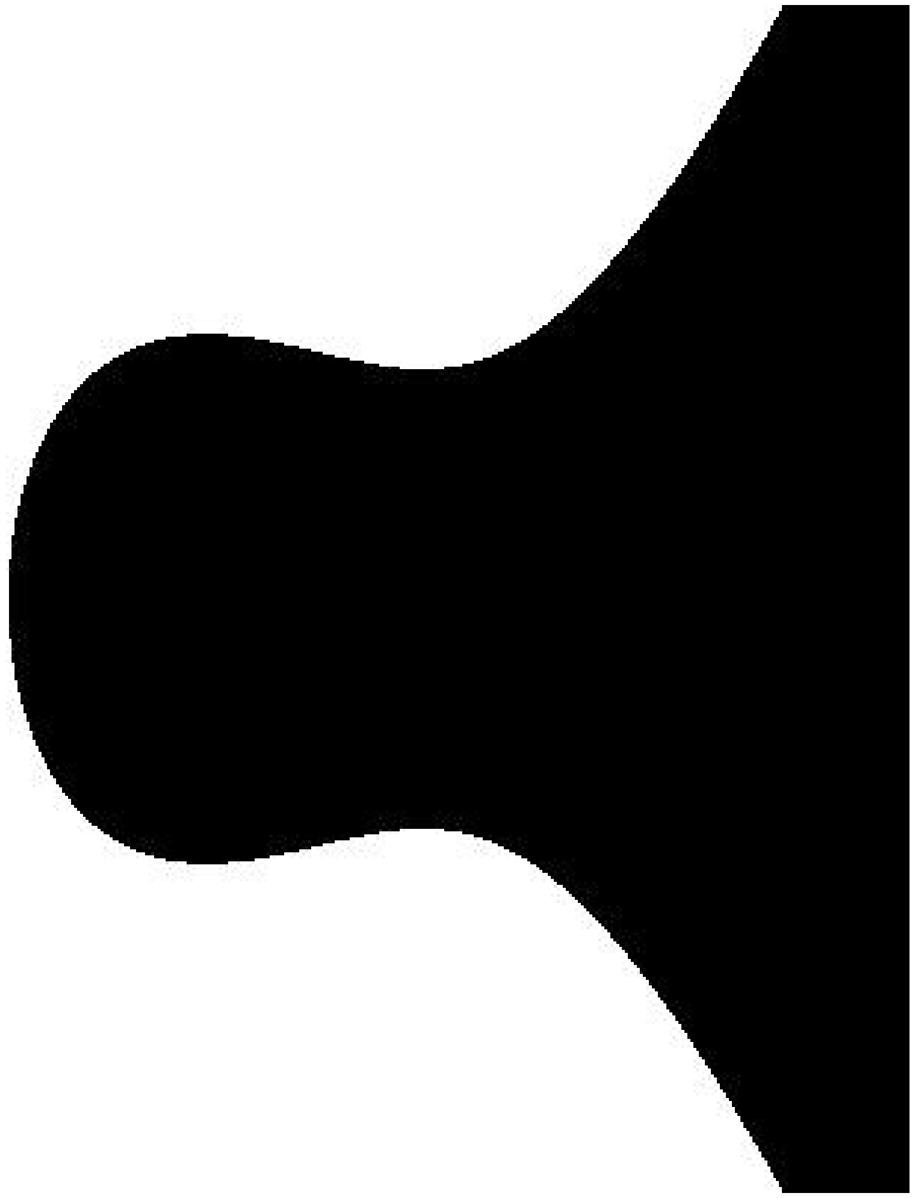

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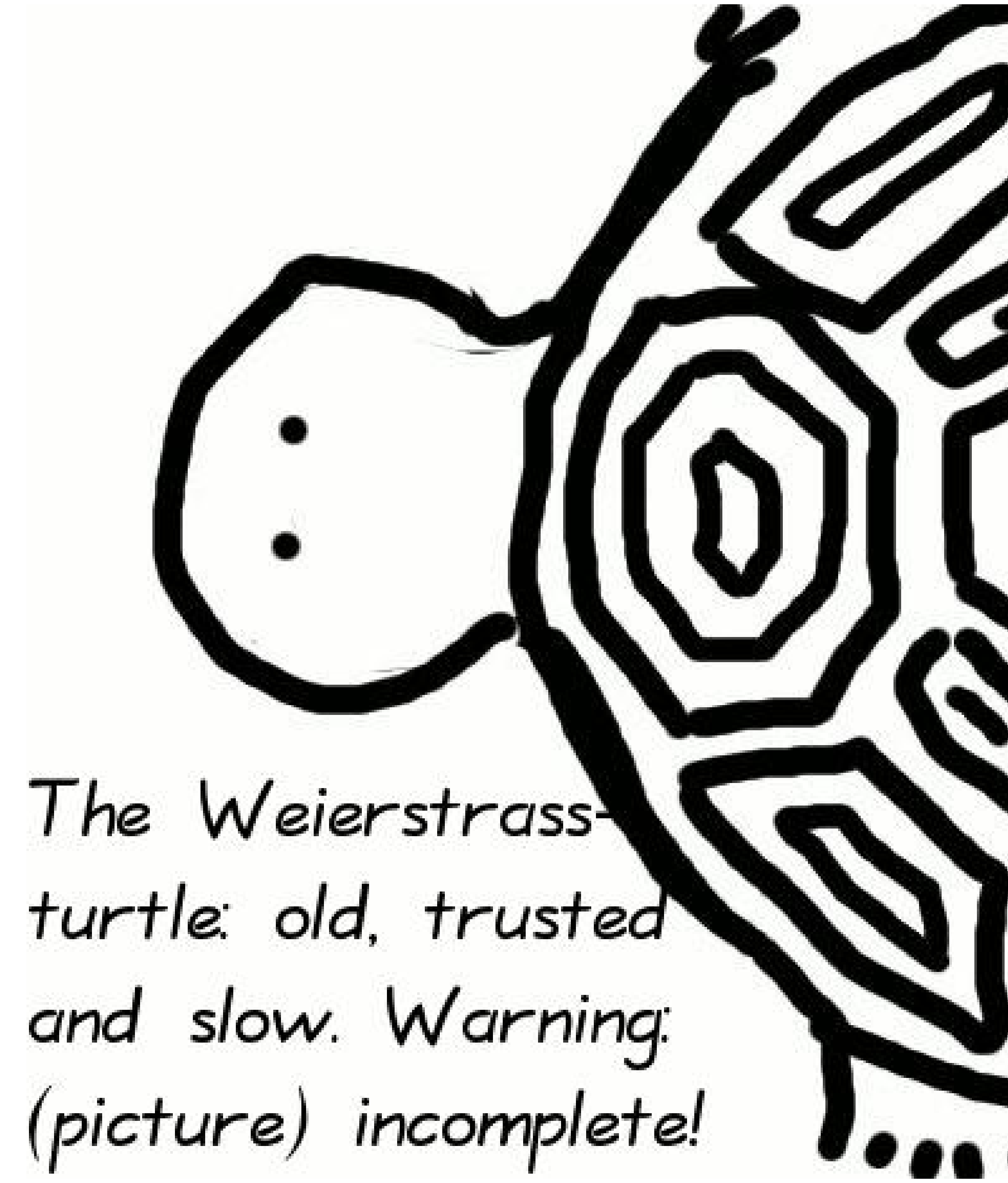

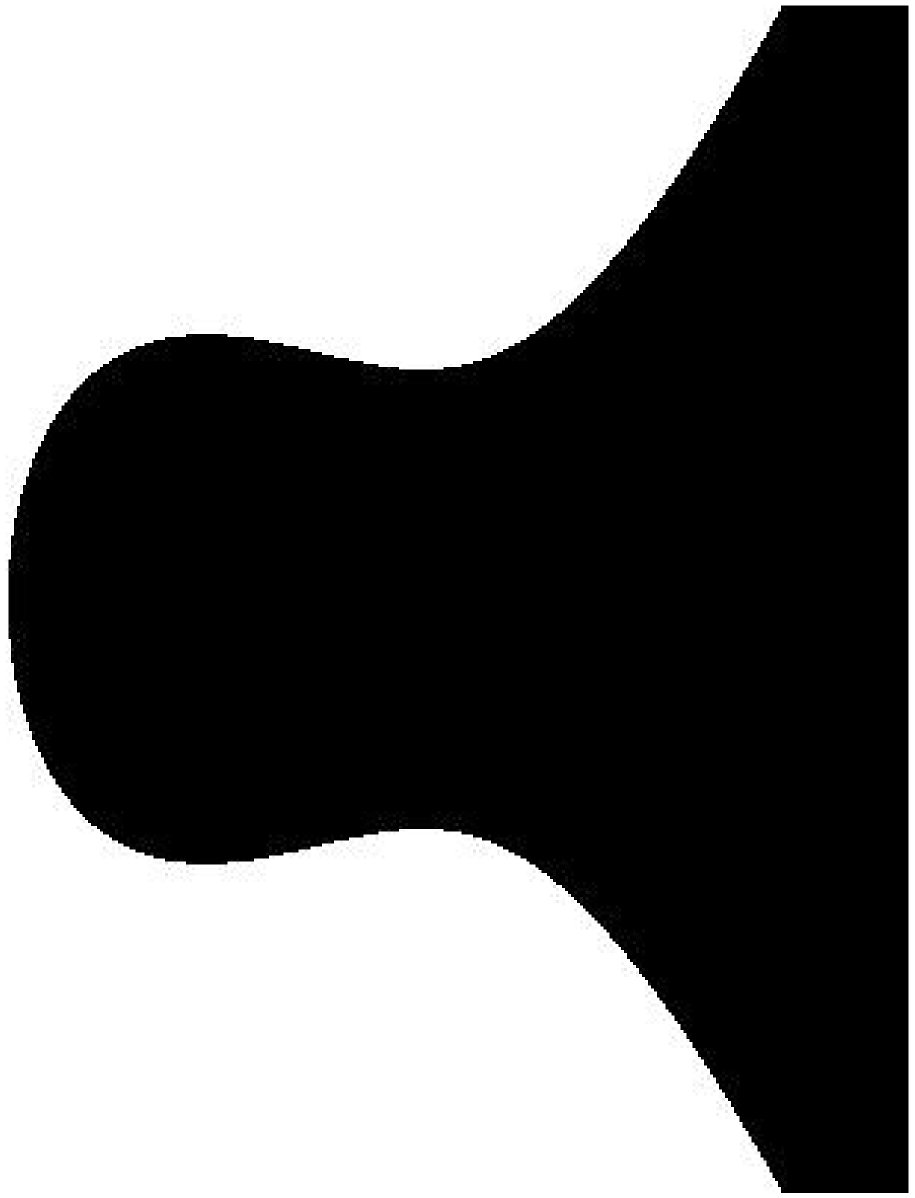

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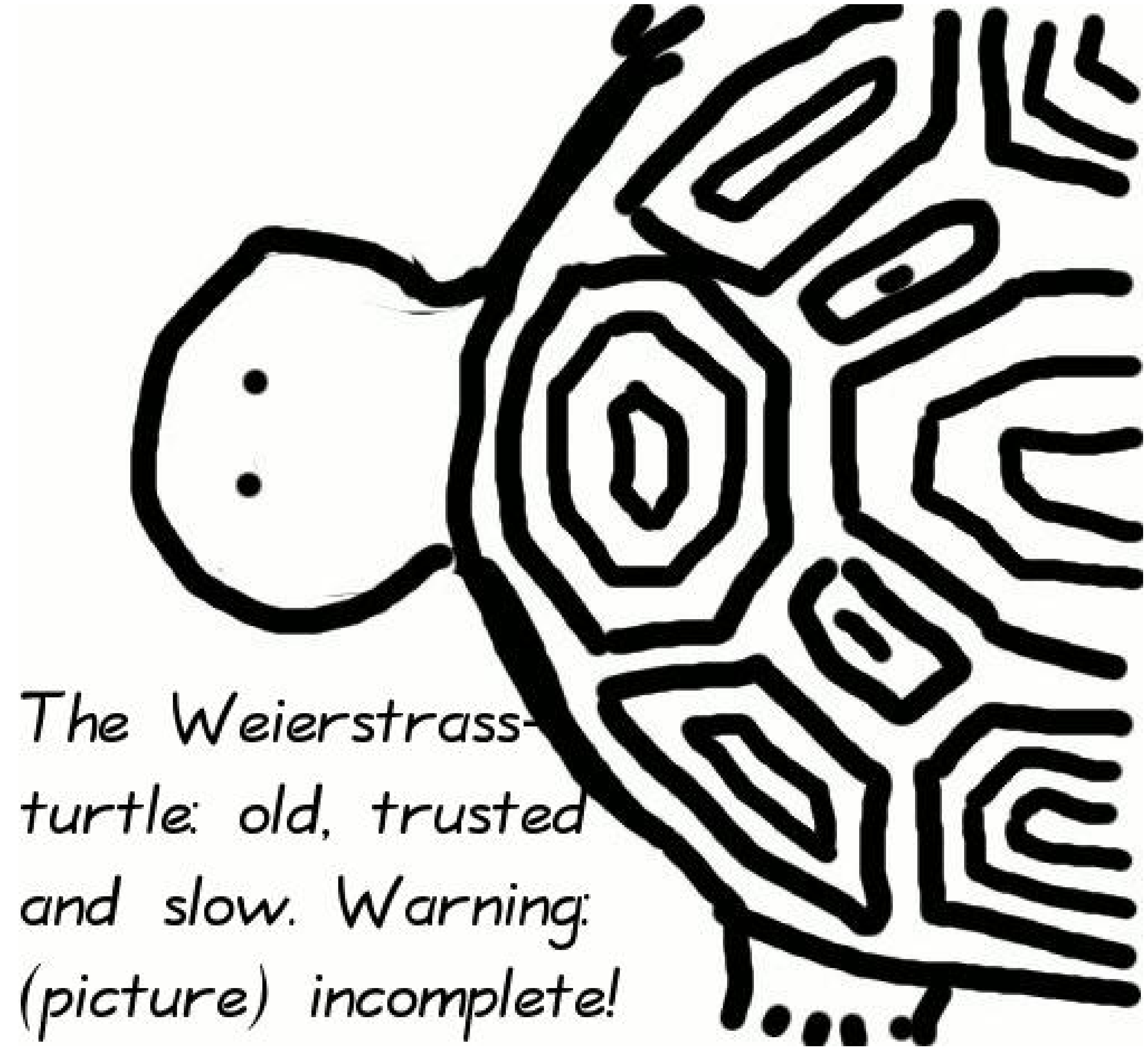
The Weierstrass-
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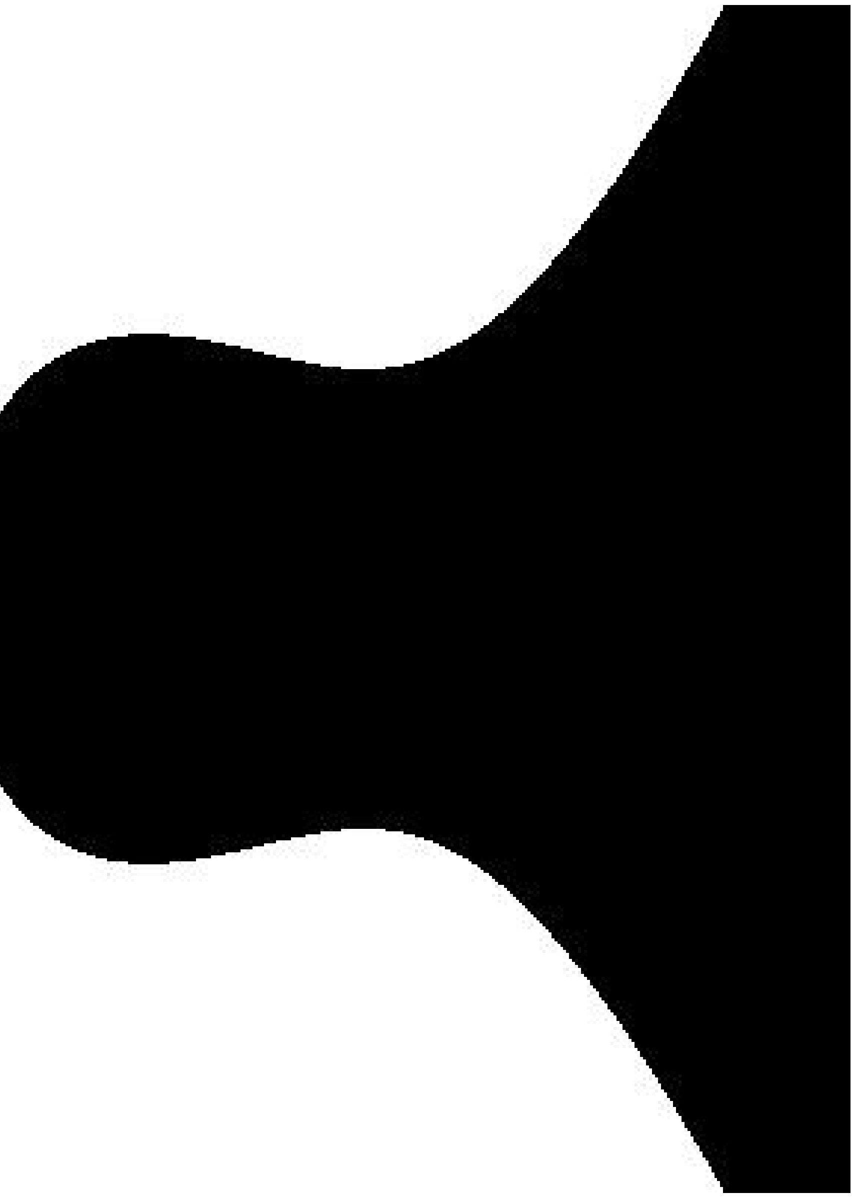
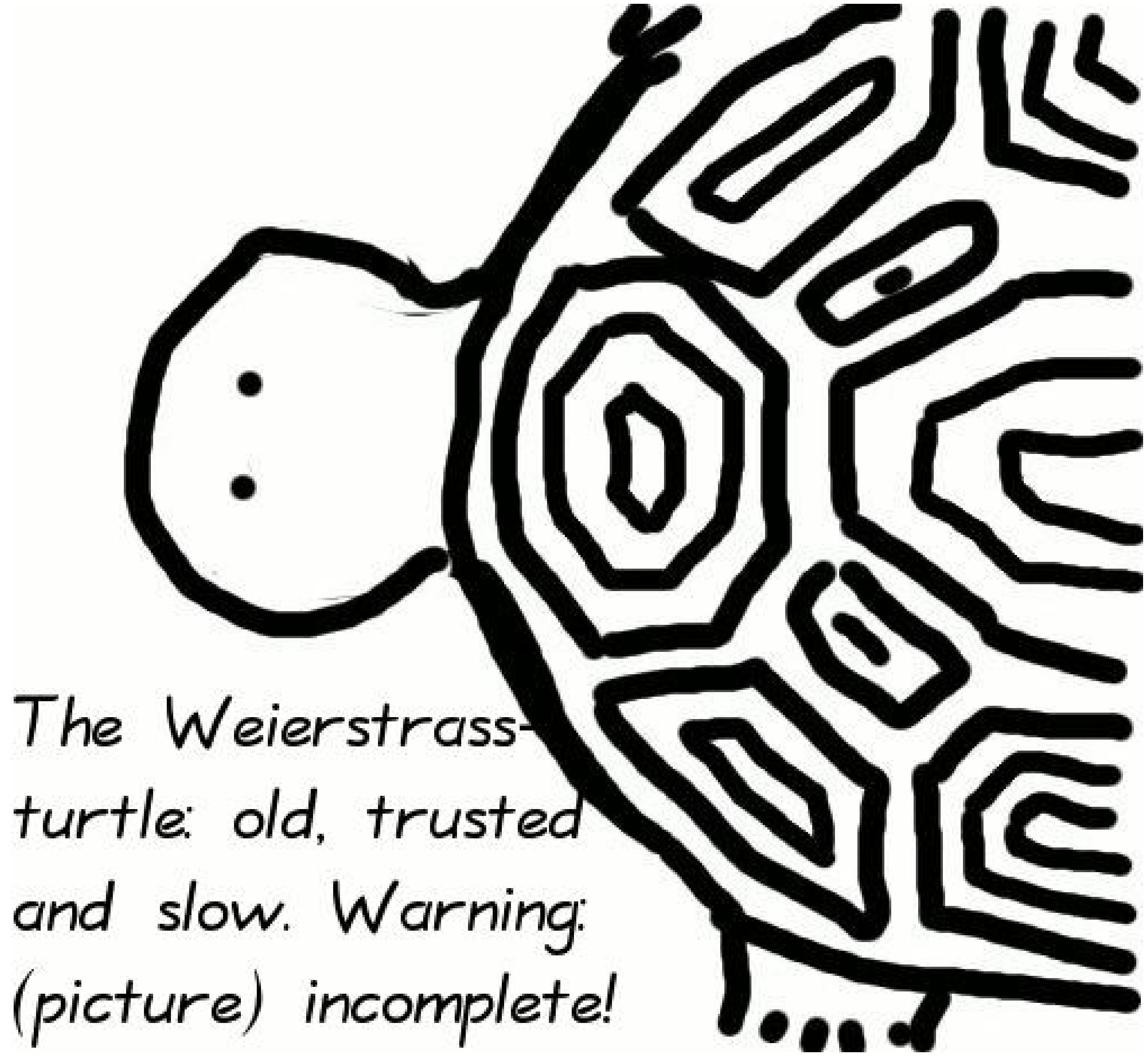


$- 0.4x + 0.7$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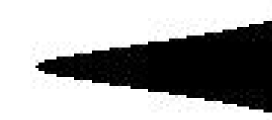
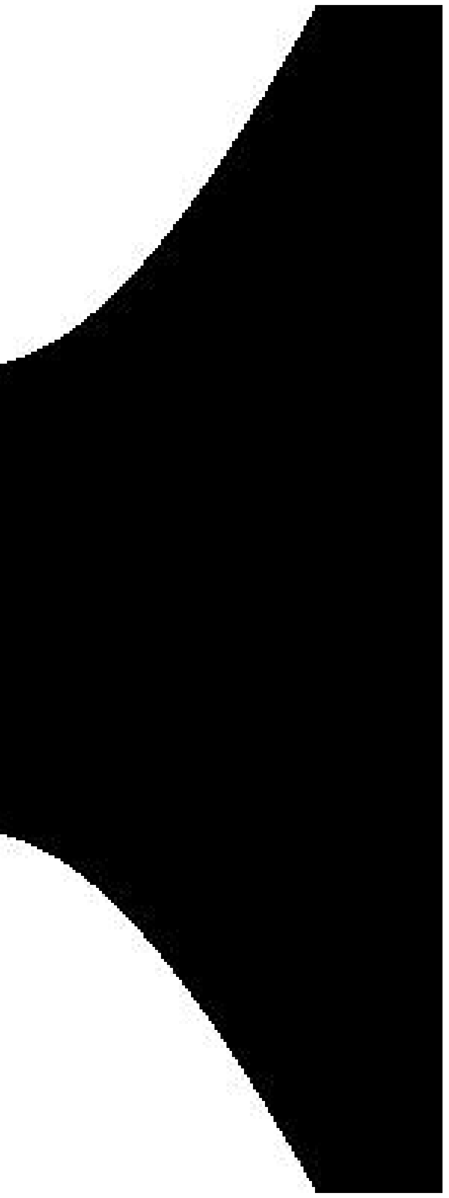


$x^2 + y^2$



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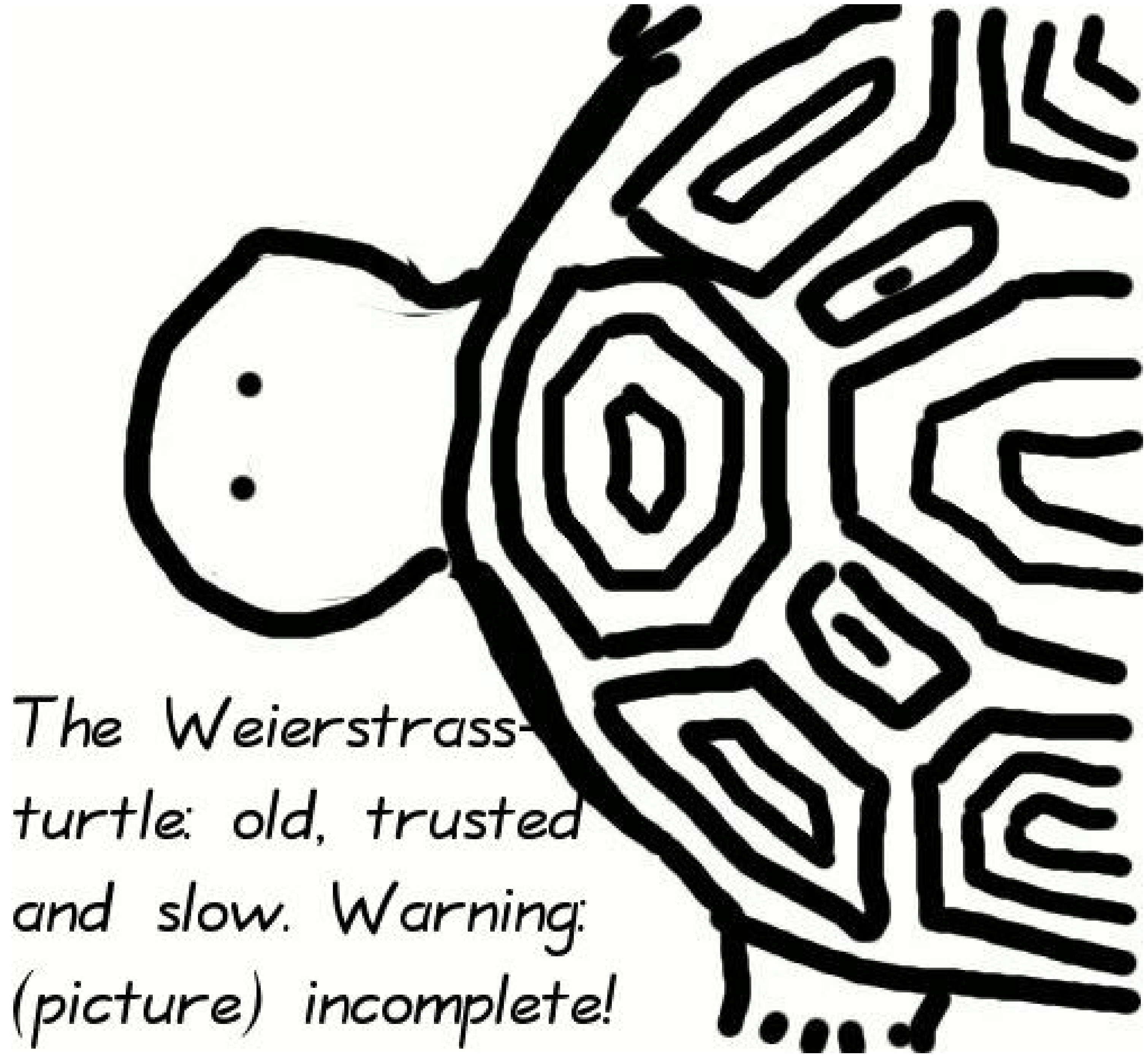
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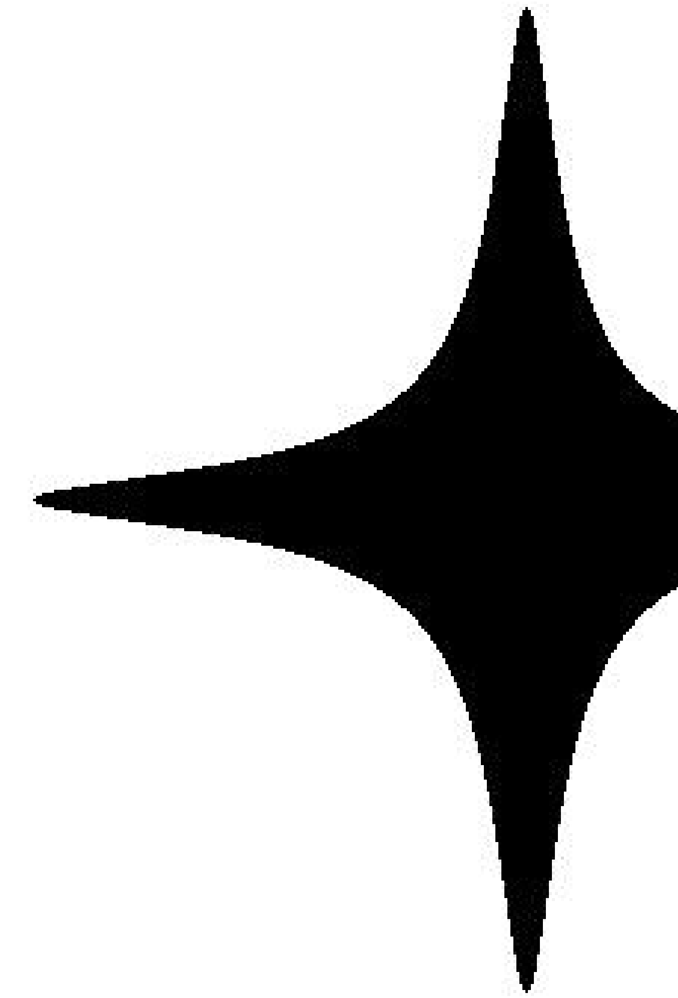
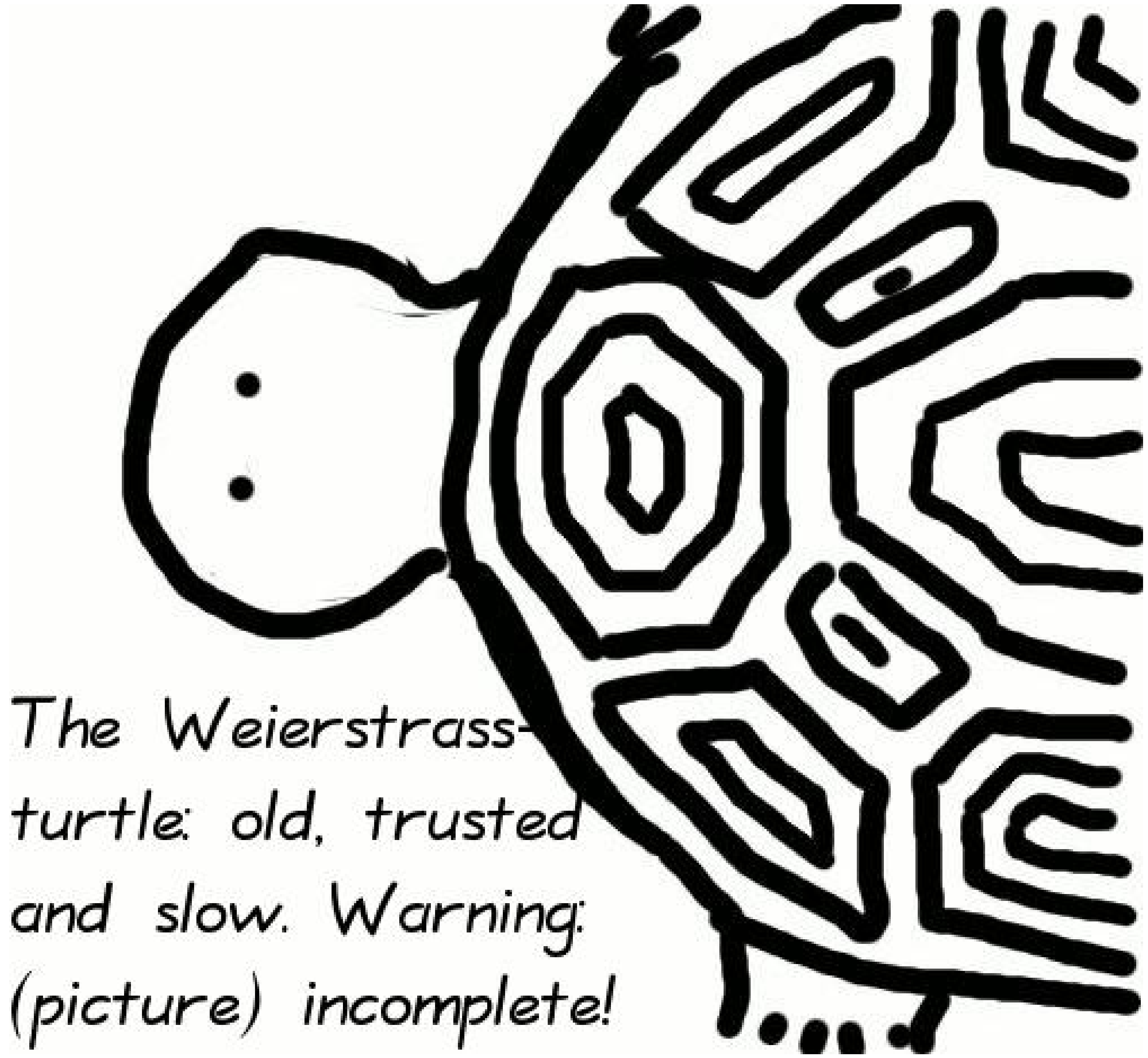
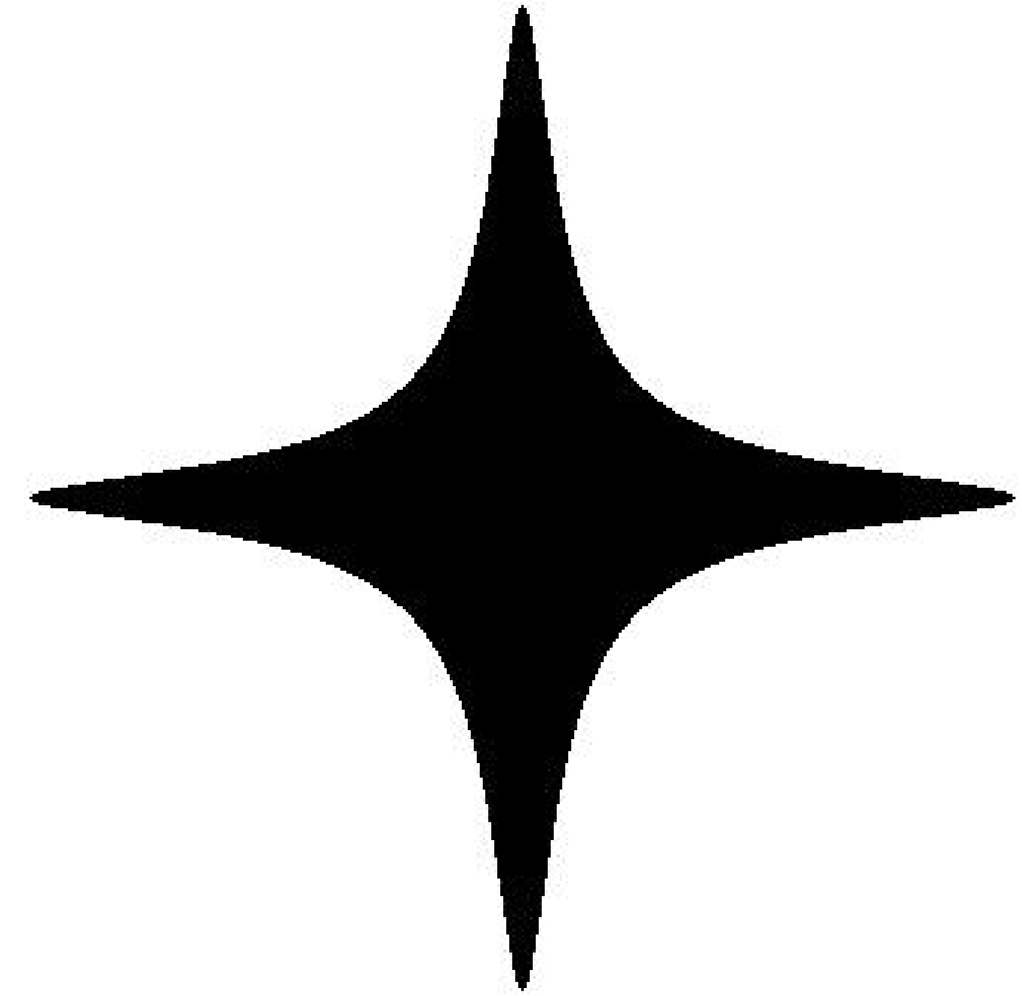
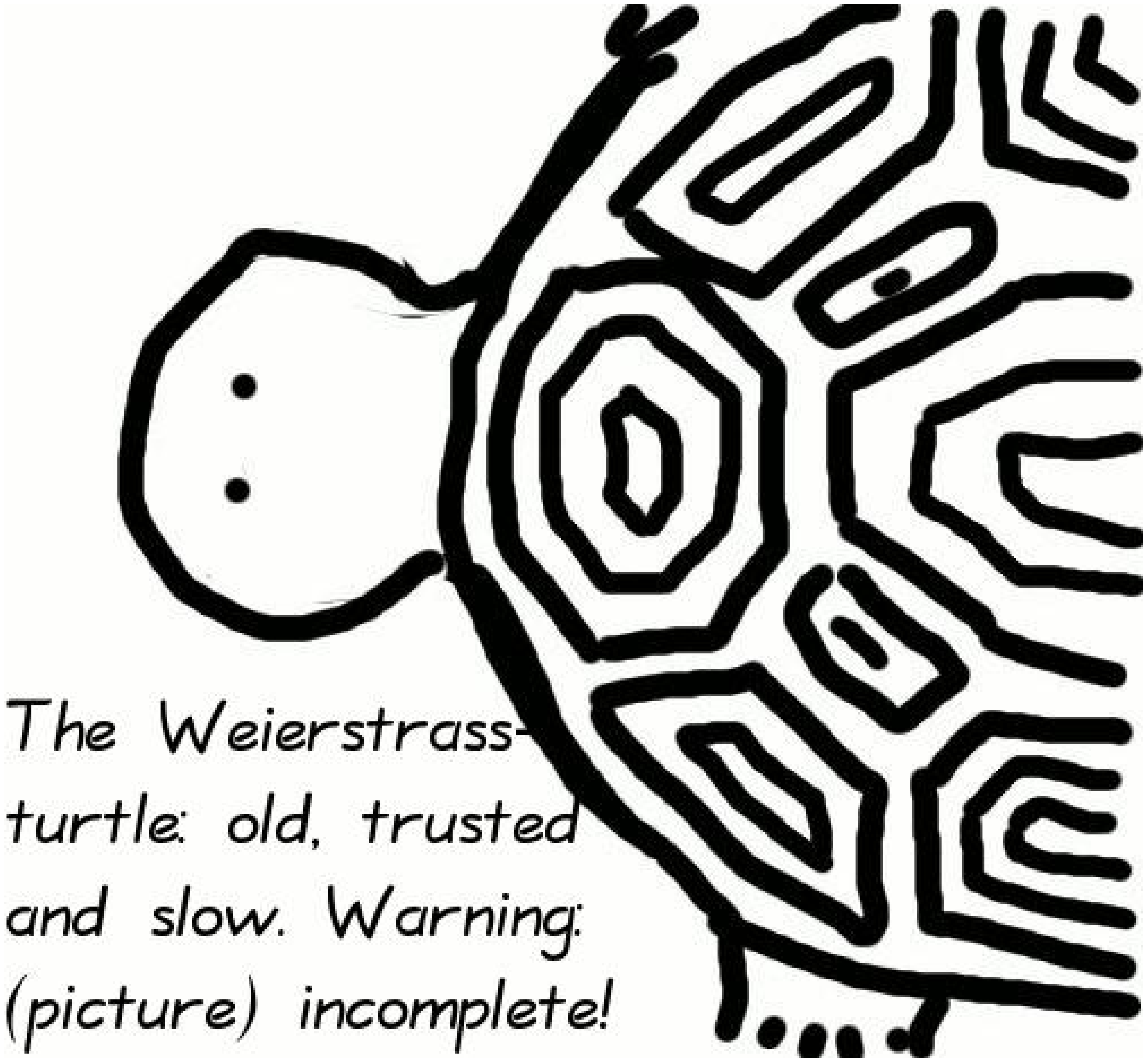


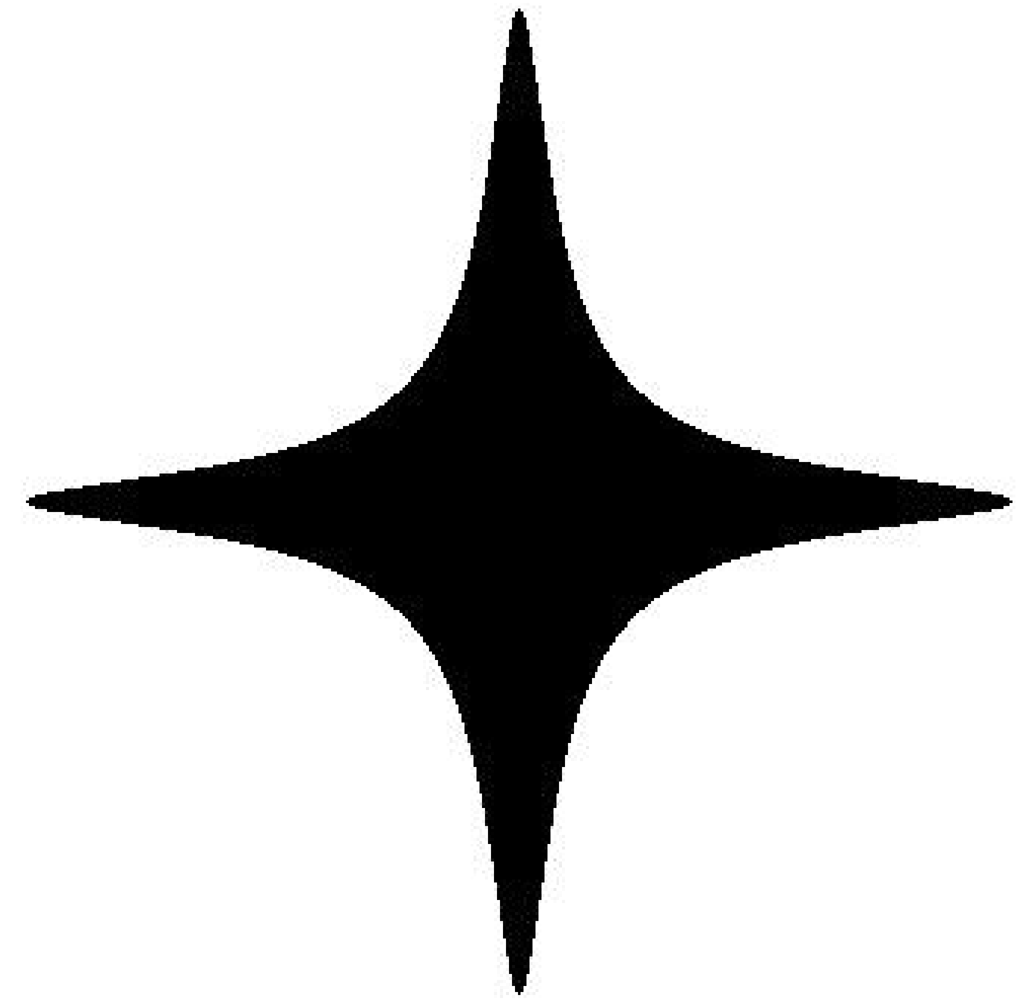
$x^2 + y^2 = 1 - 300$





$$x^2 + y^2 = 1 - 300x^2y^2$$





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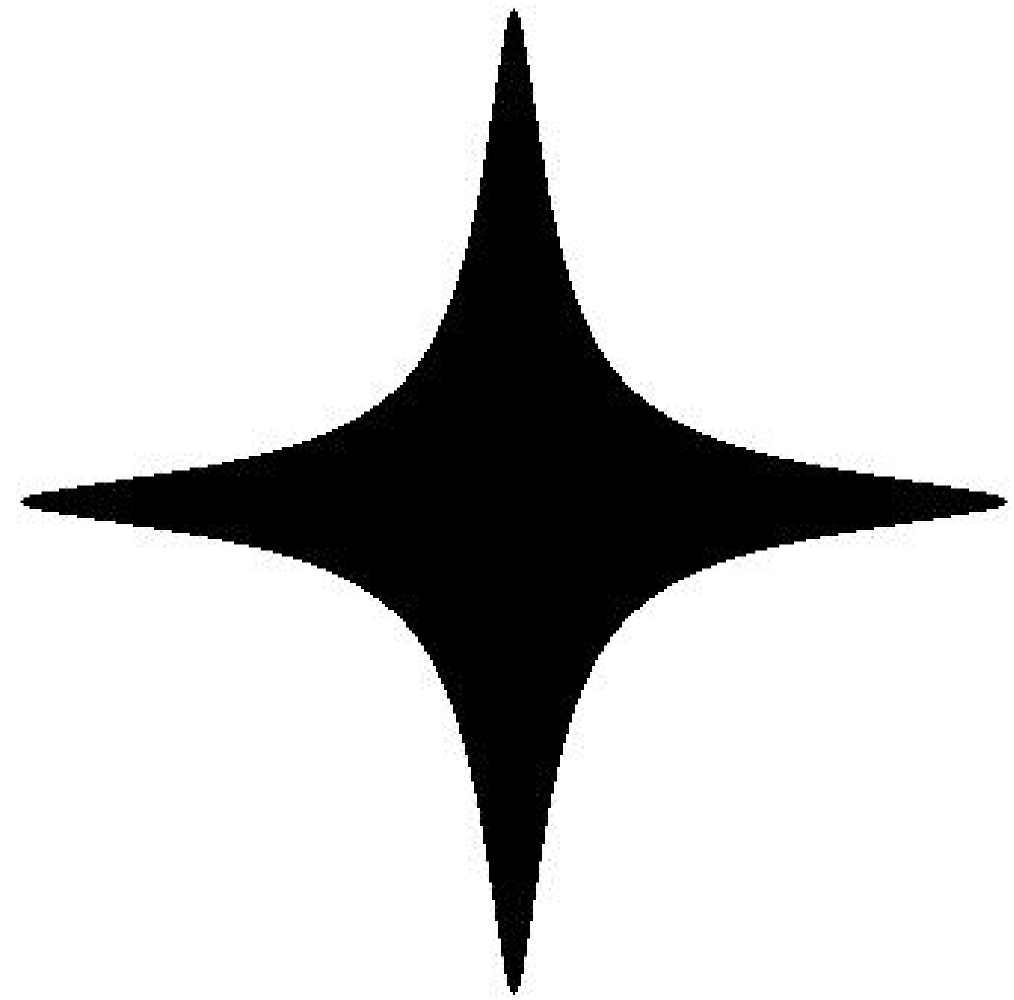

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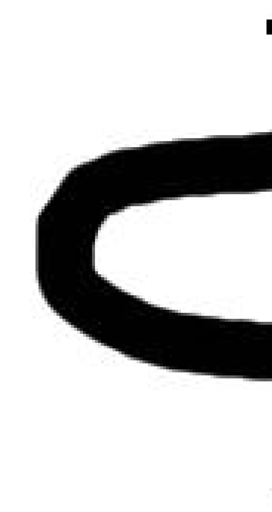

The Edv
starfish:
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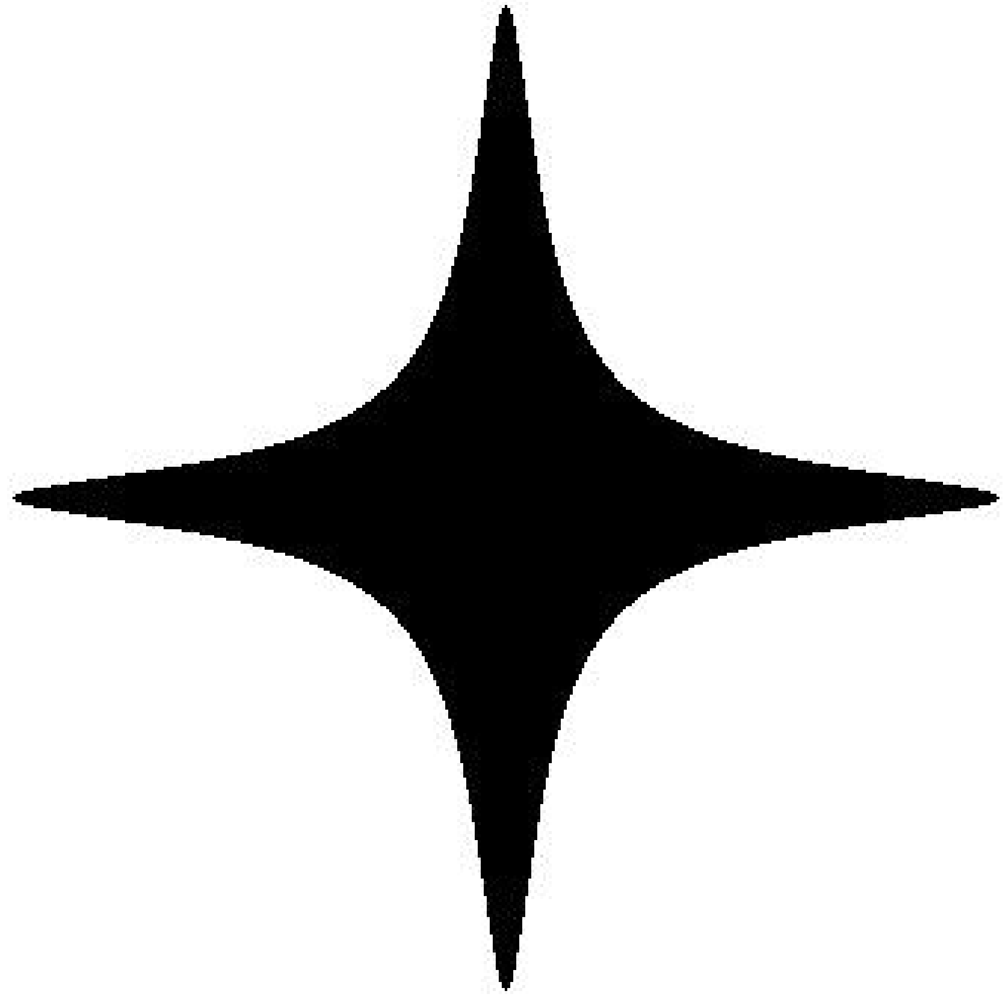

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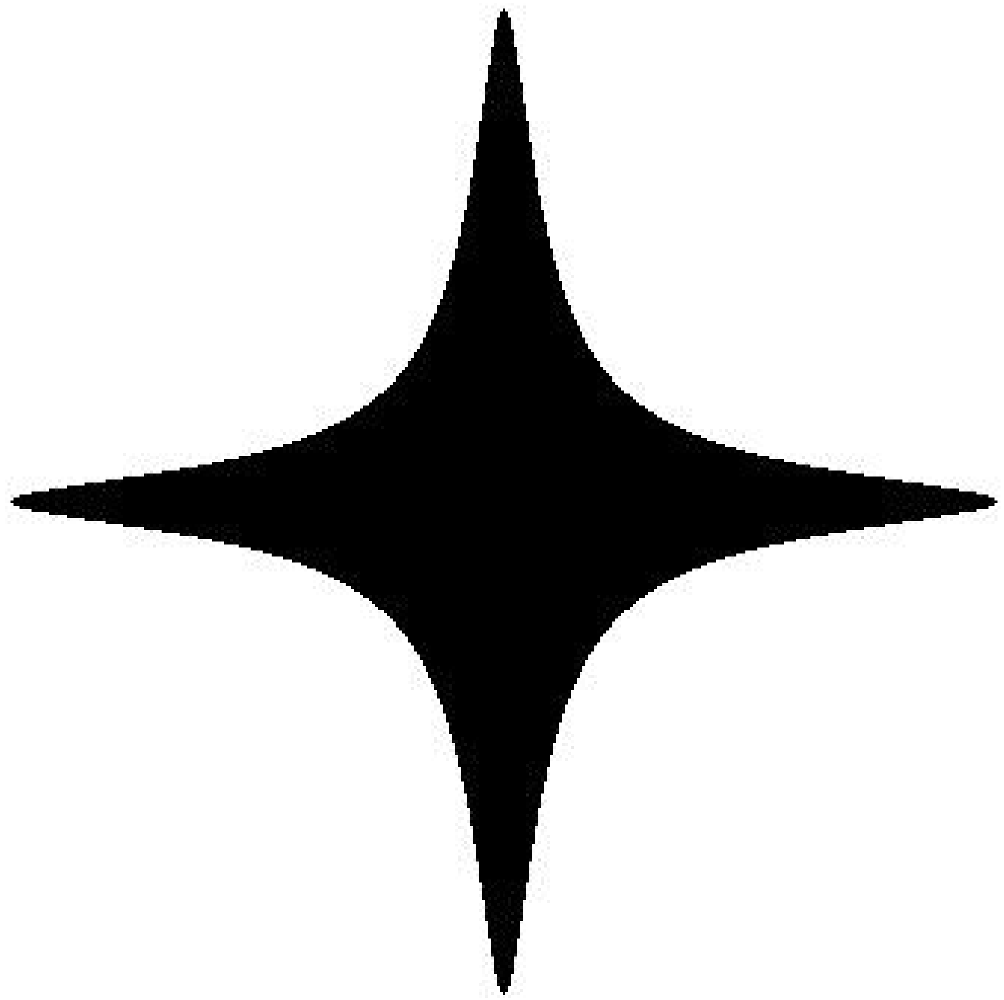

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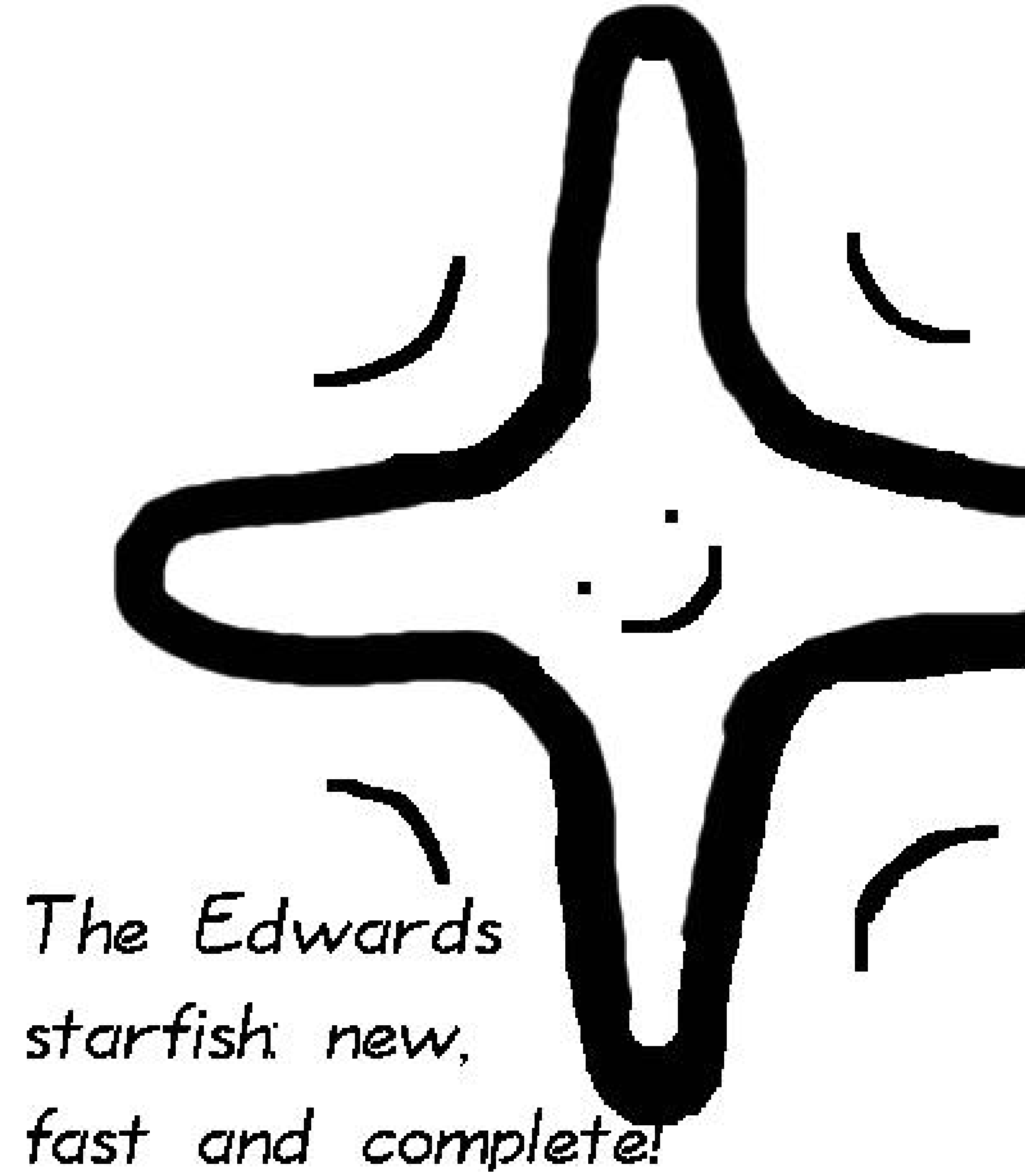

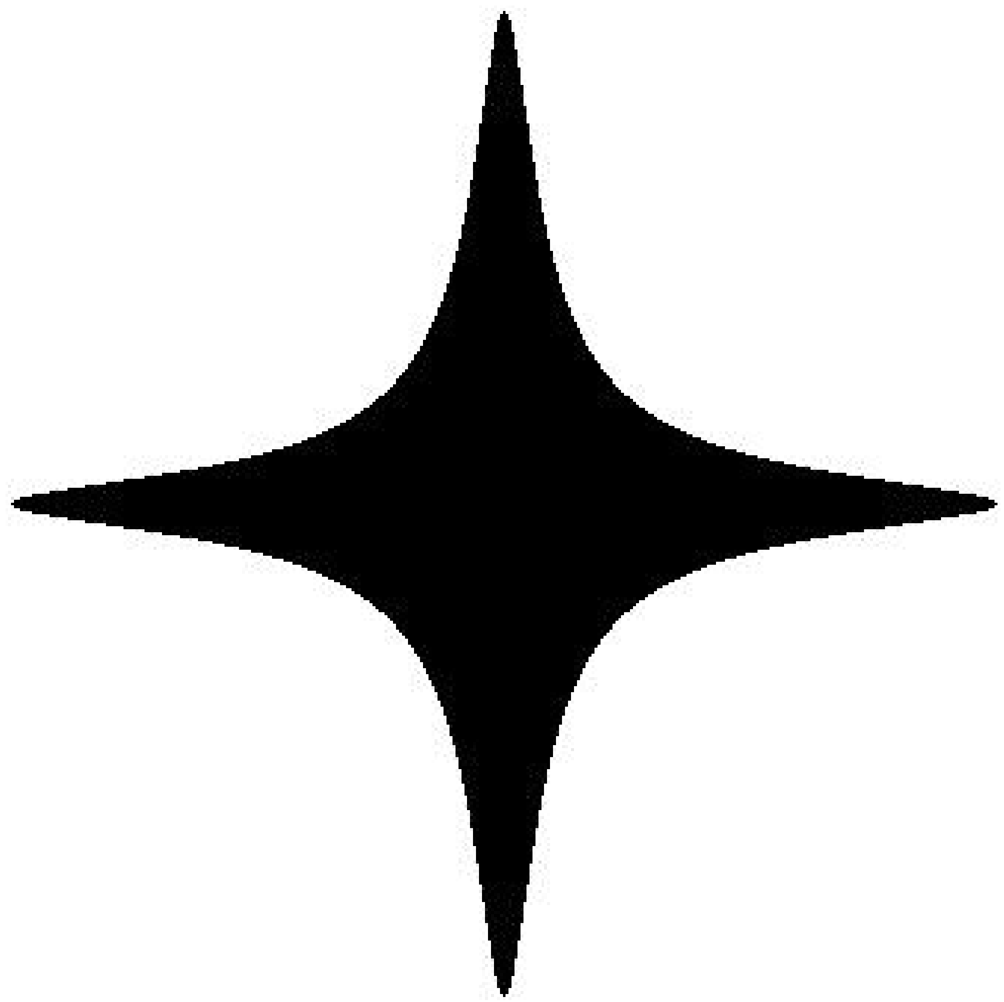

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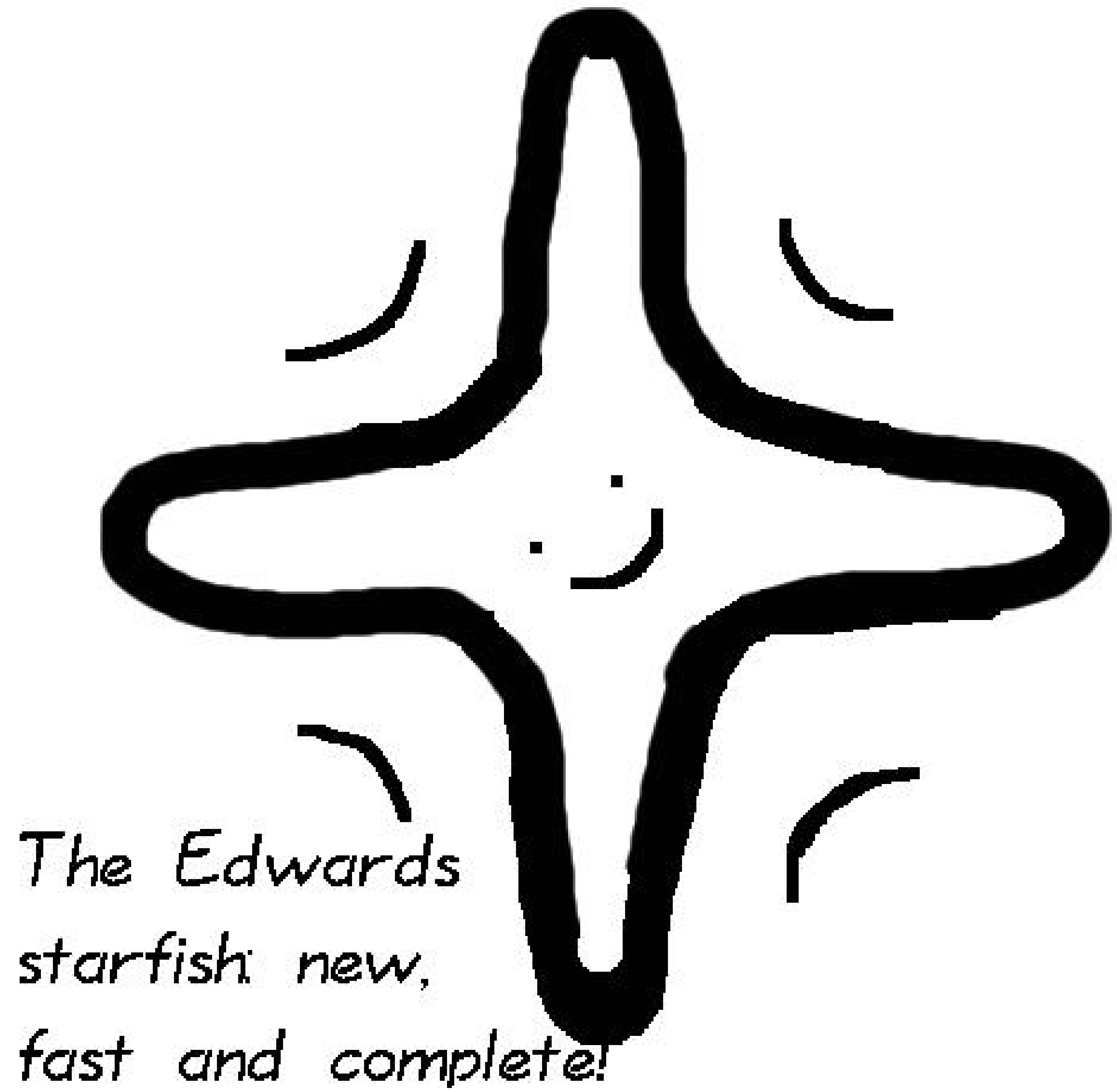
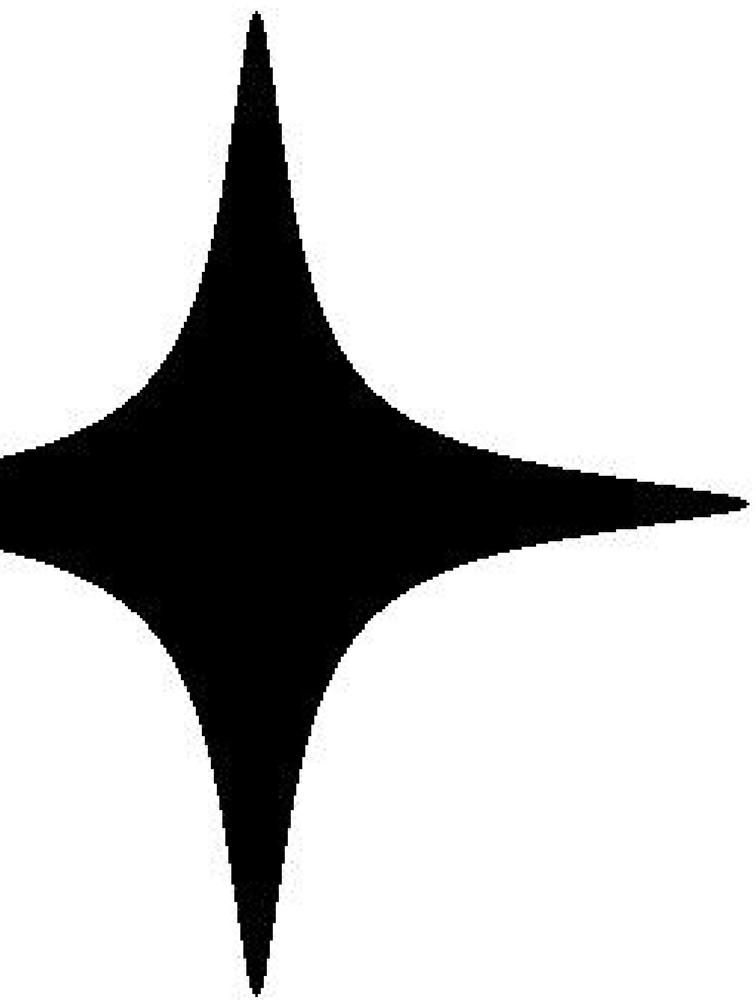
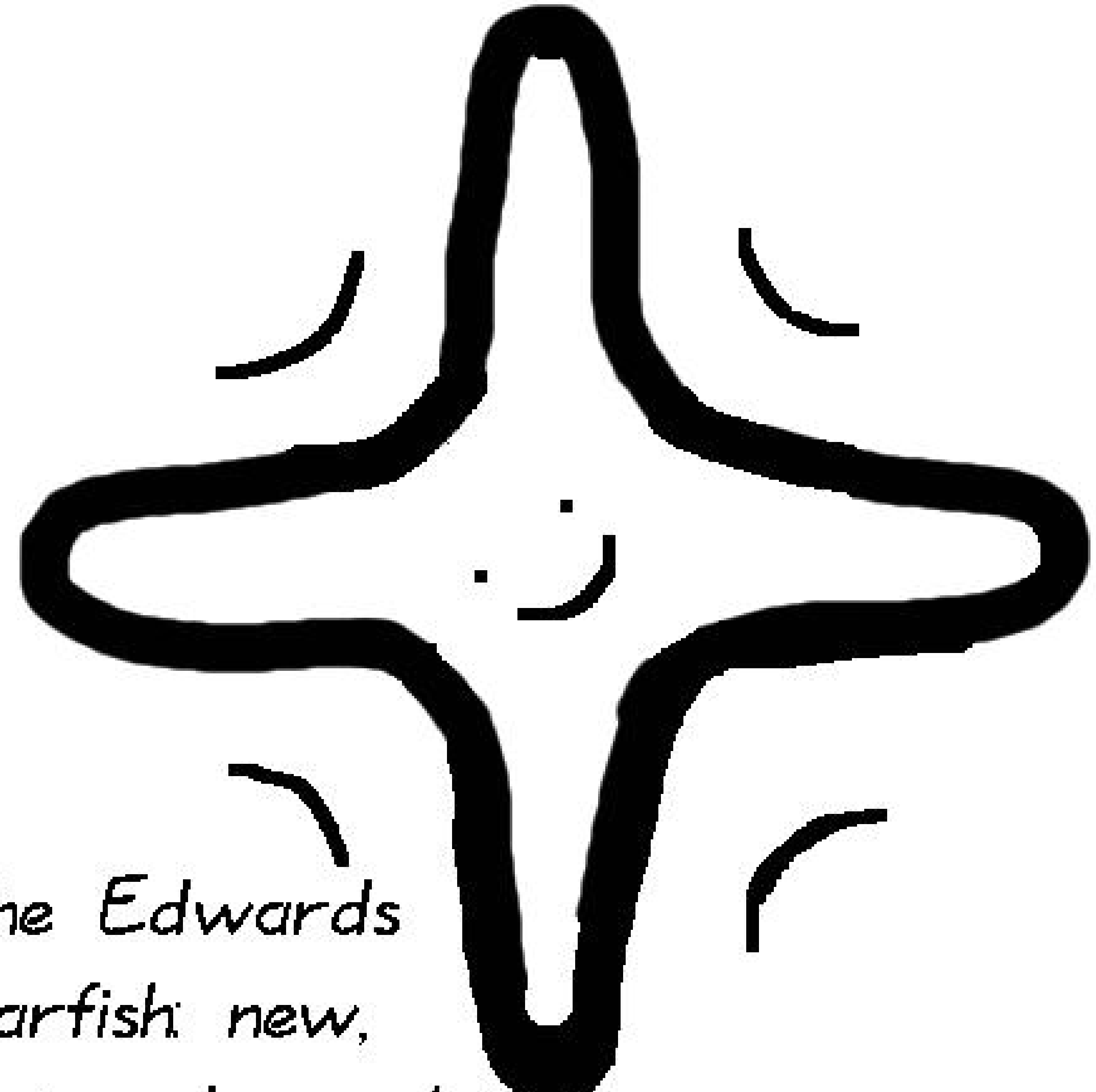


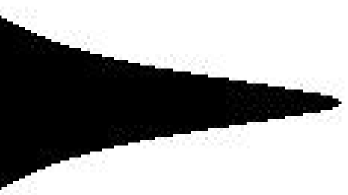


$= 1 - 300x^2y^2$

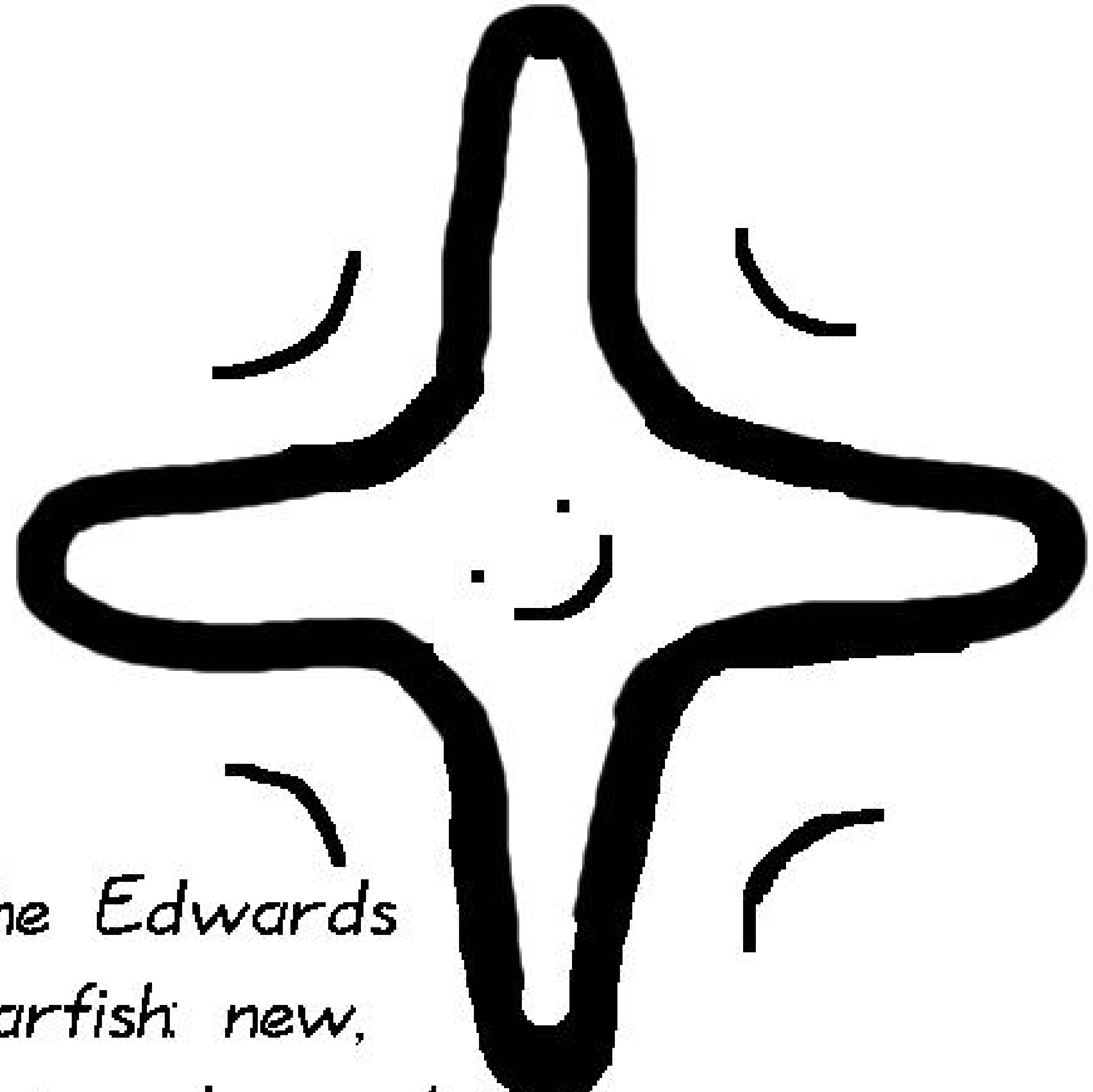


The Edwards starfish: new, fast and complete!

Start!



$0x^2y^2$



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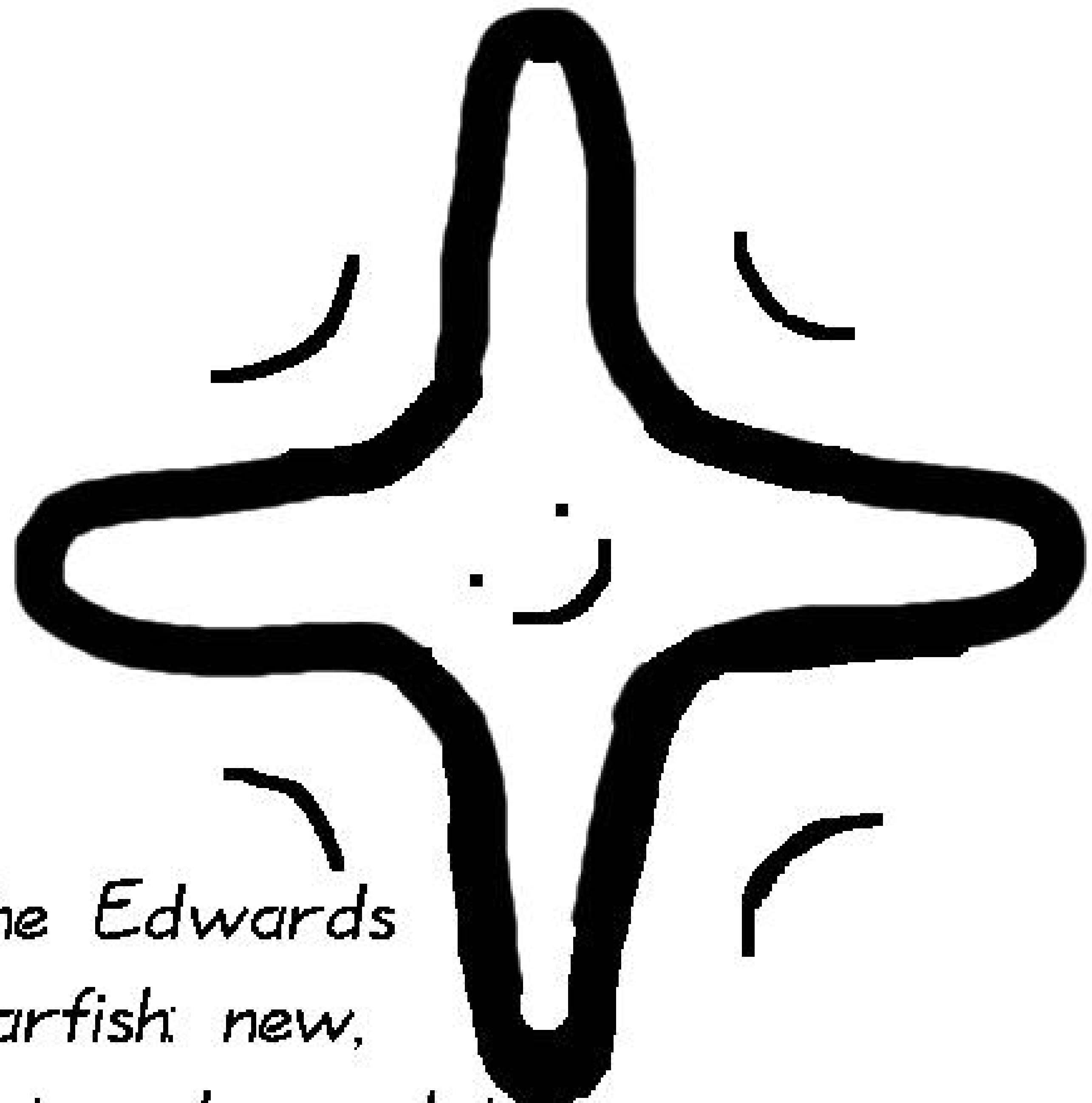

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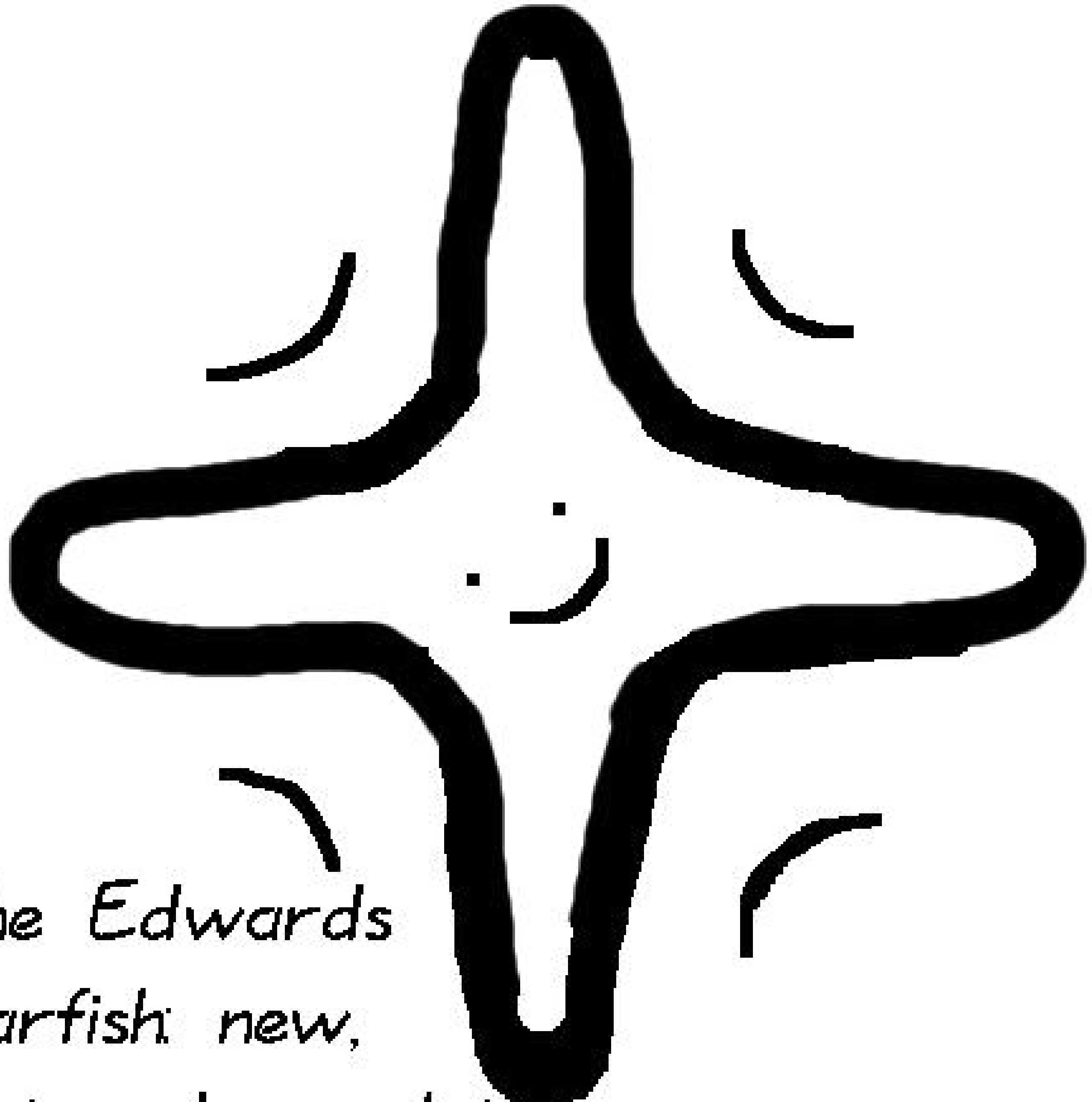

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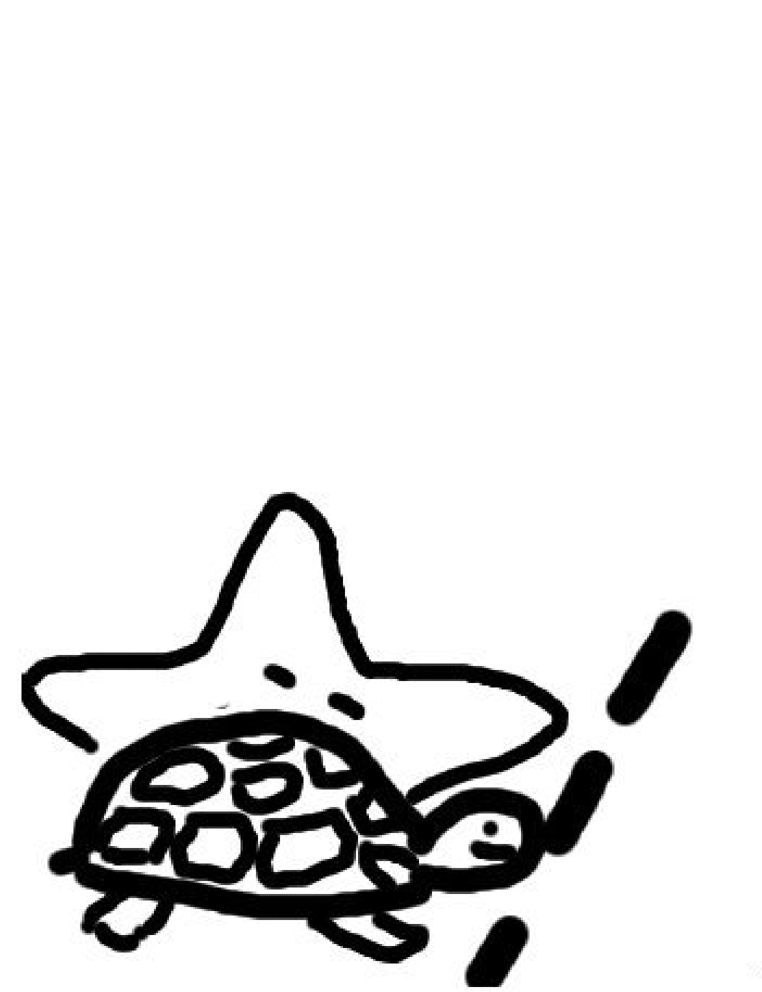

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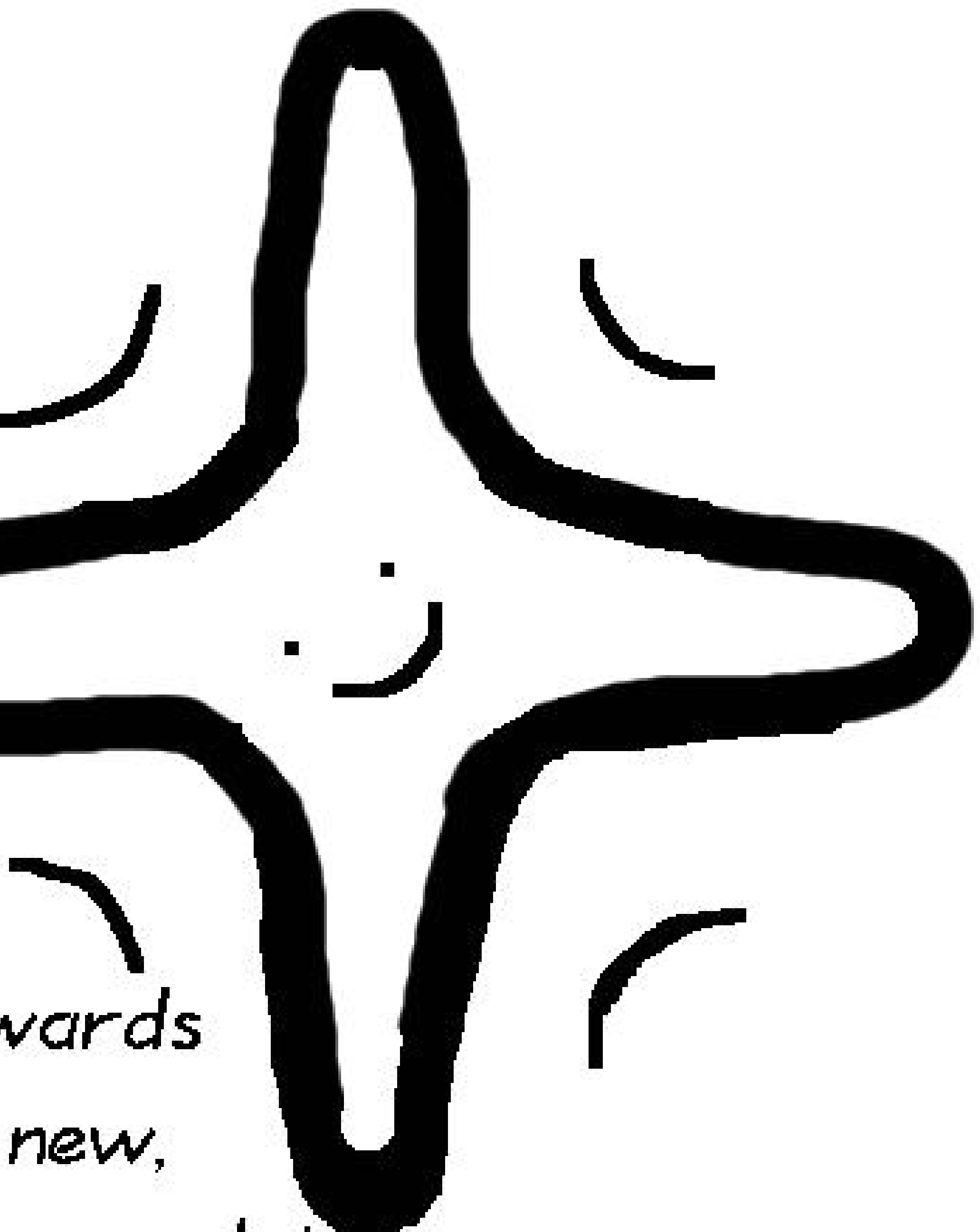

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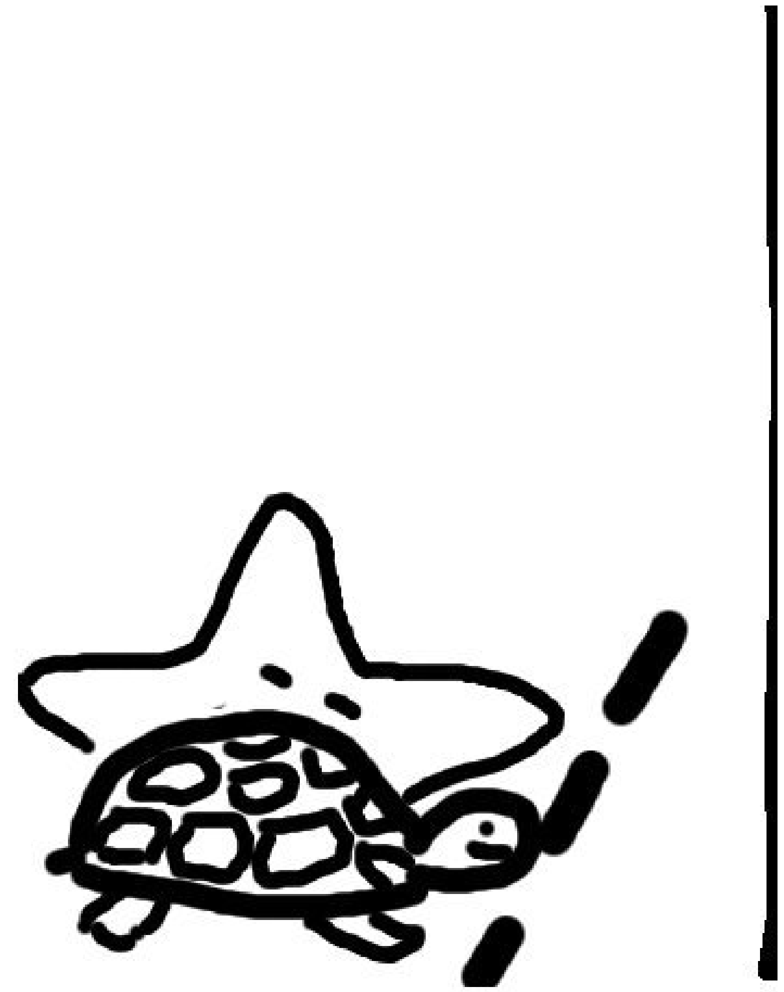

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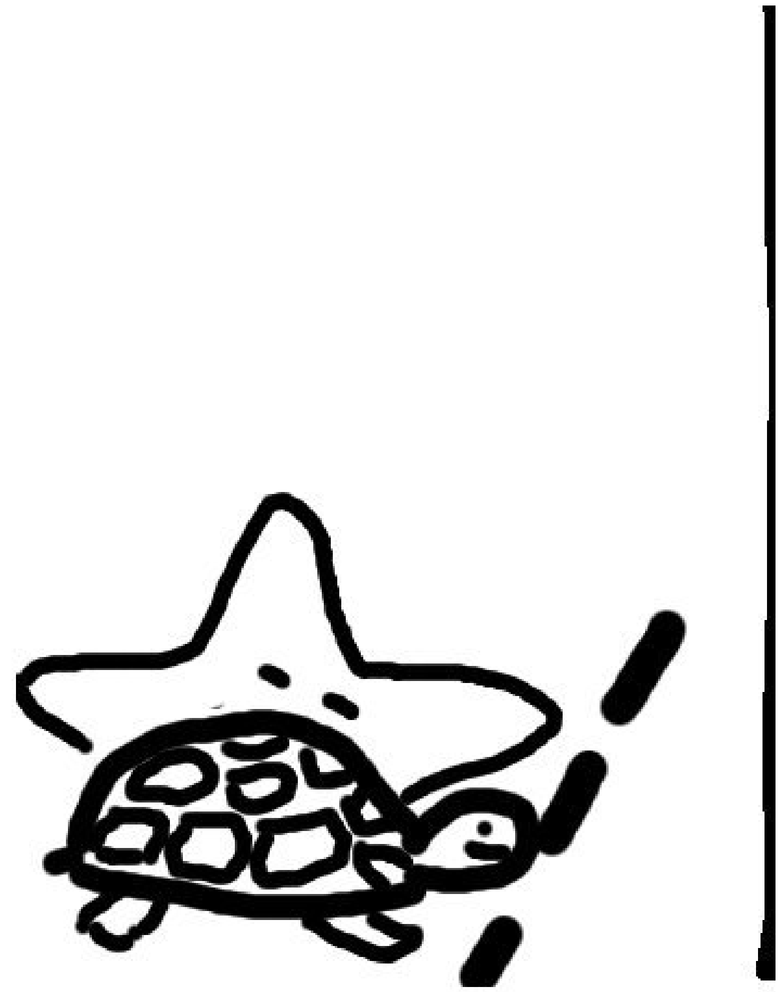
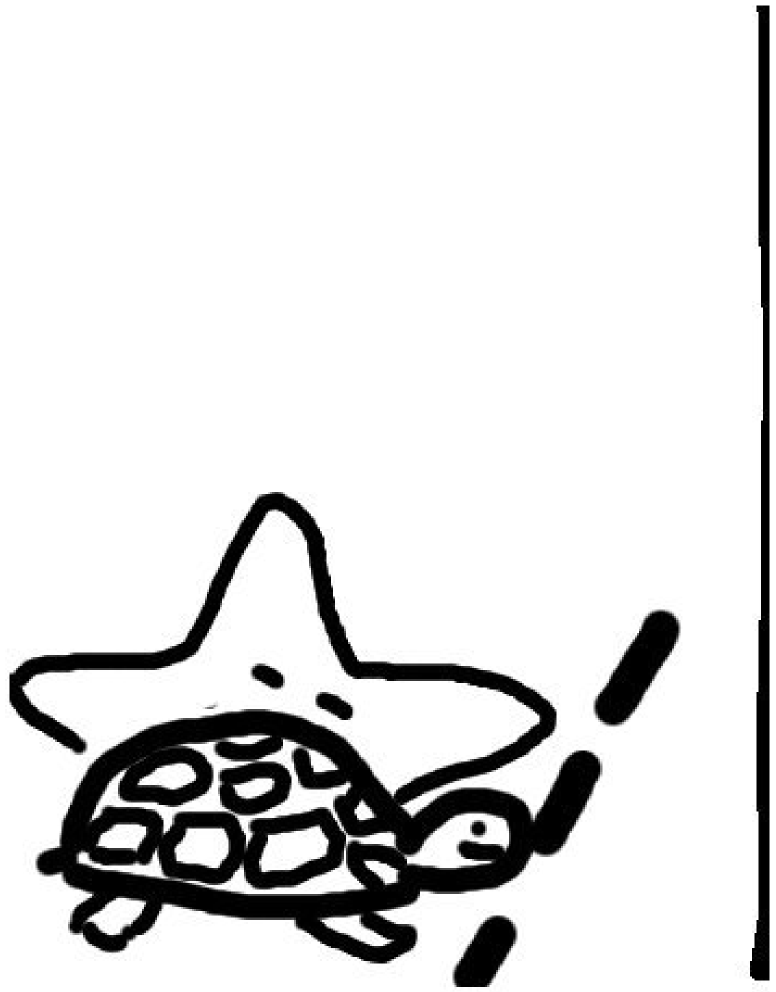


Start!

Weierstrass sets off; Ed
left behind sleeping



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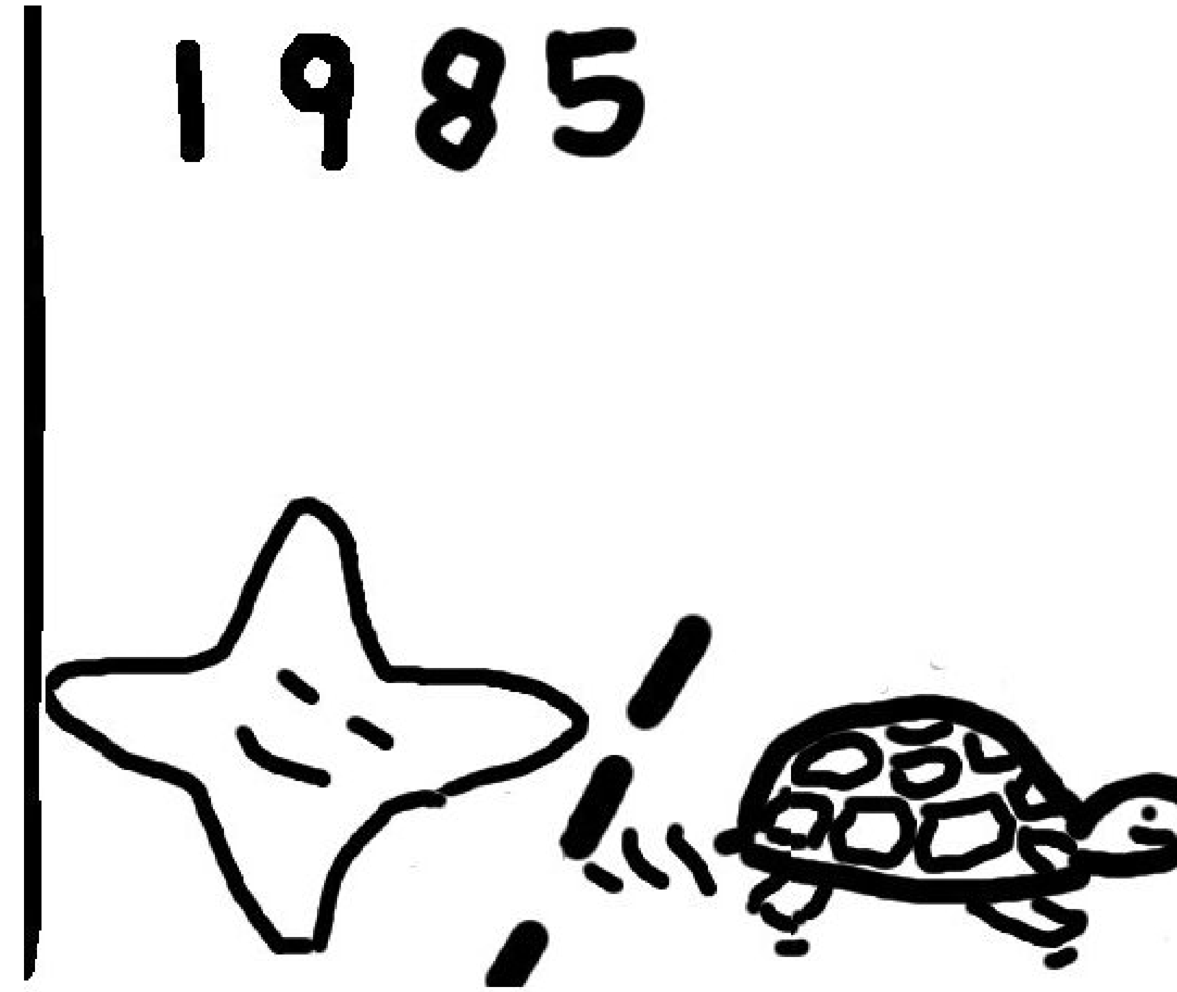

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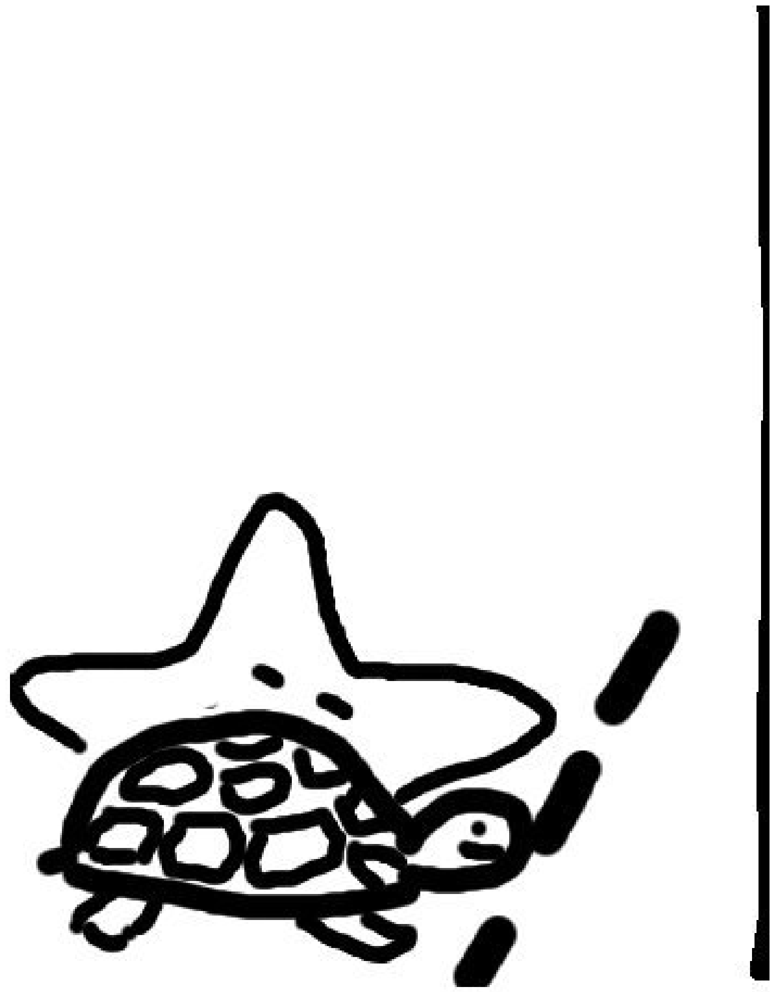
left behind sleeping



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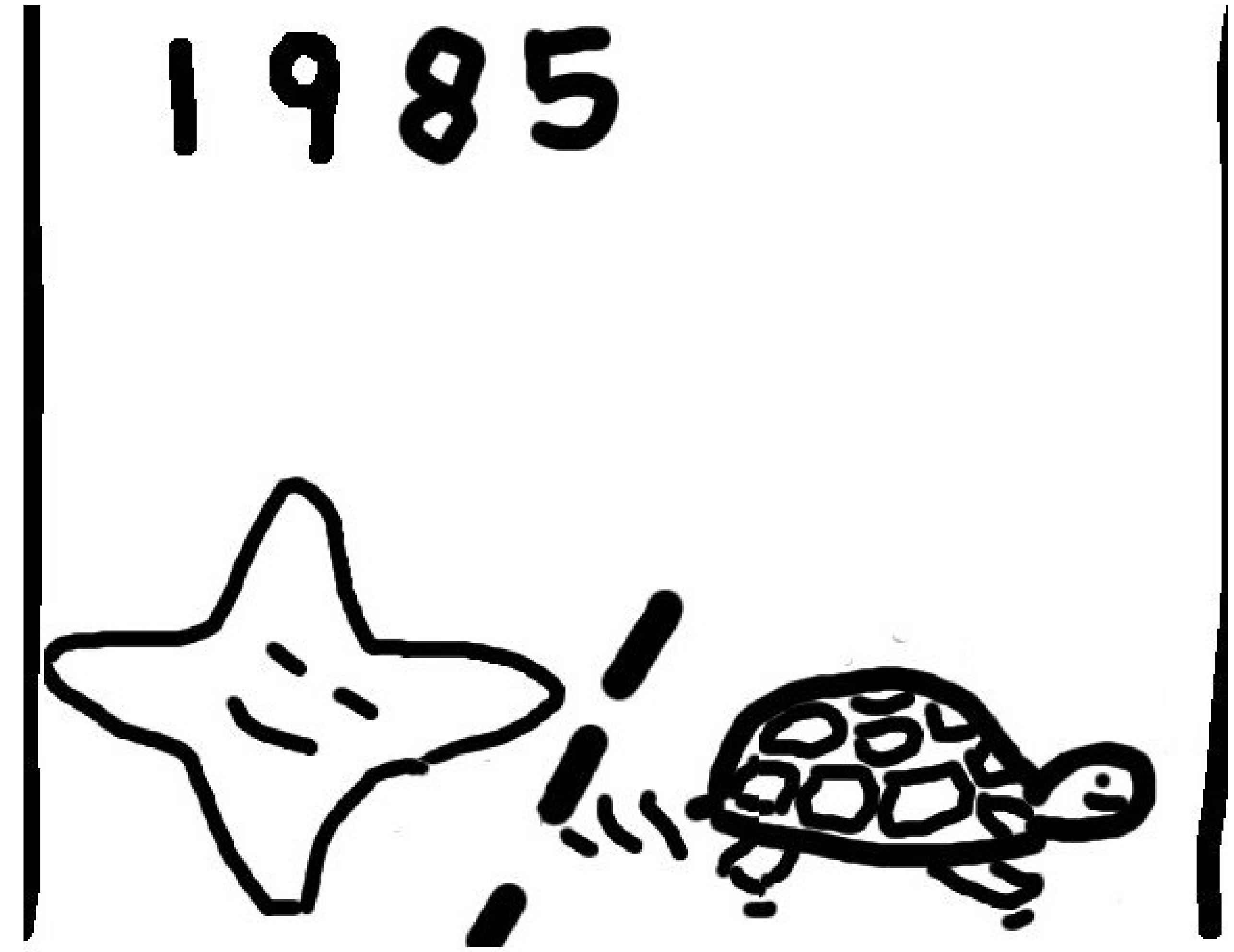

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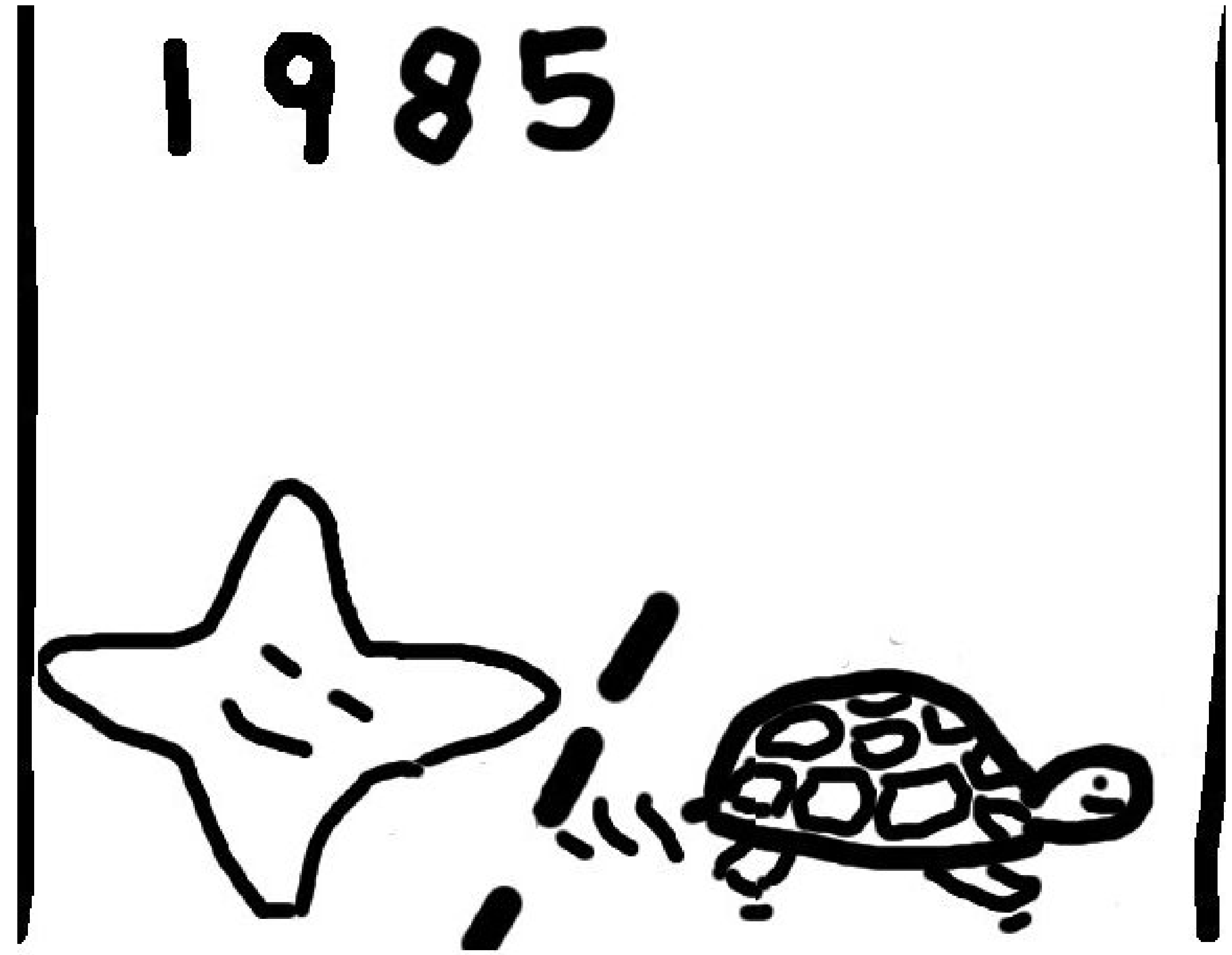

Weierstrass sets off, Edwards
left behind sleeping

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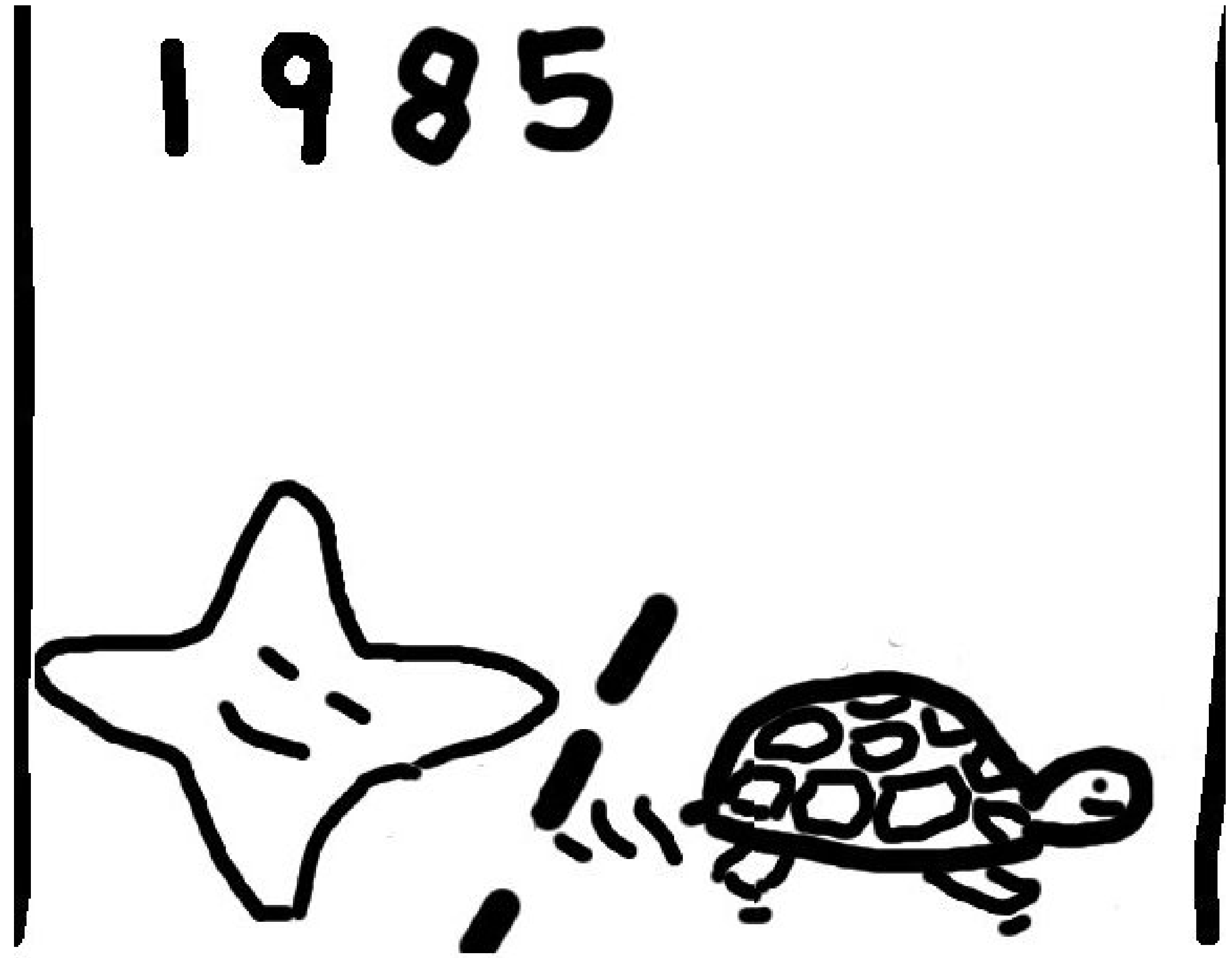

Weierstrass sets off. Edwards
left behind sleeping

Weierstrass has made sc
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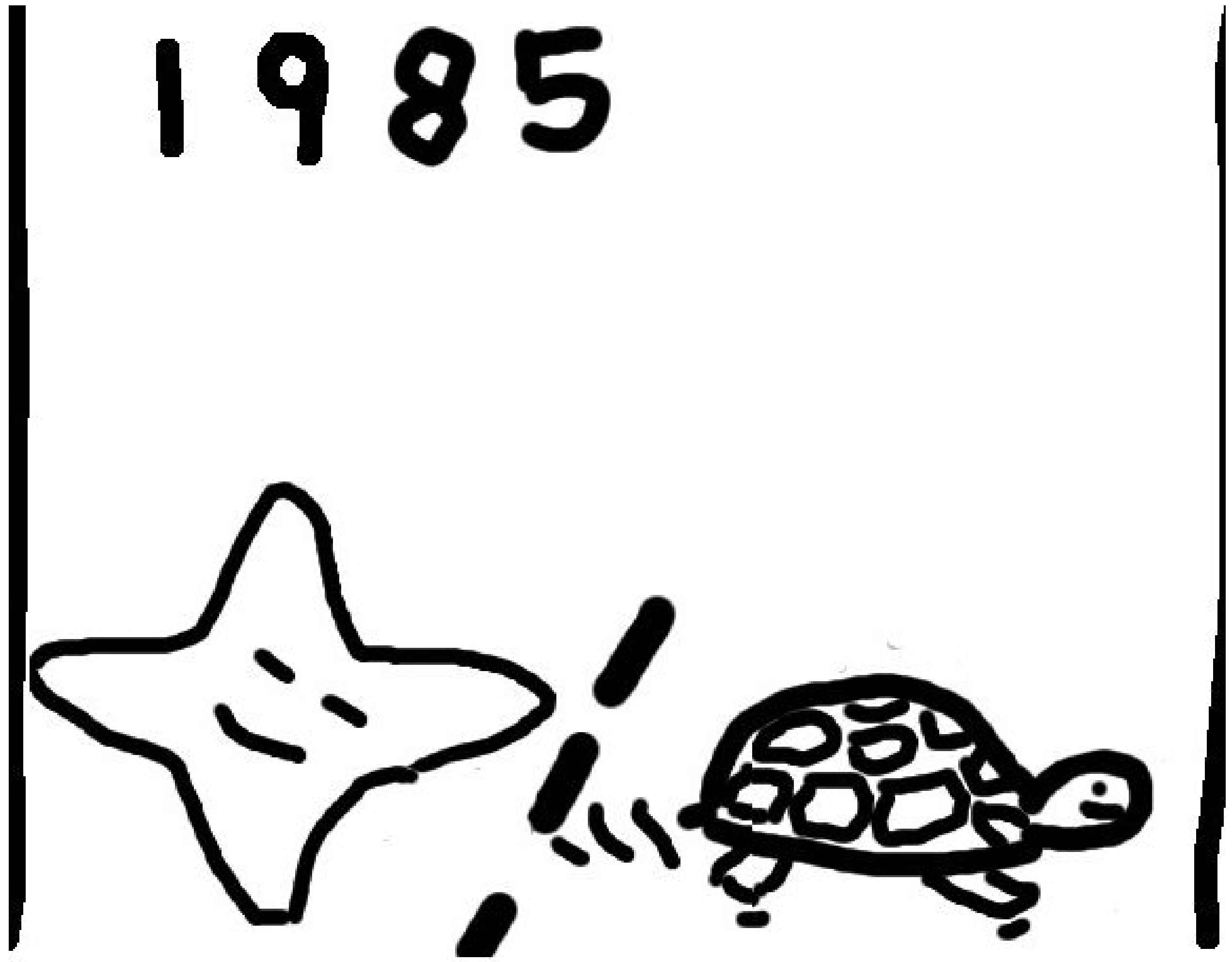
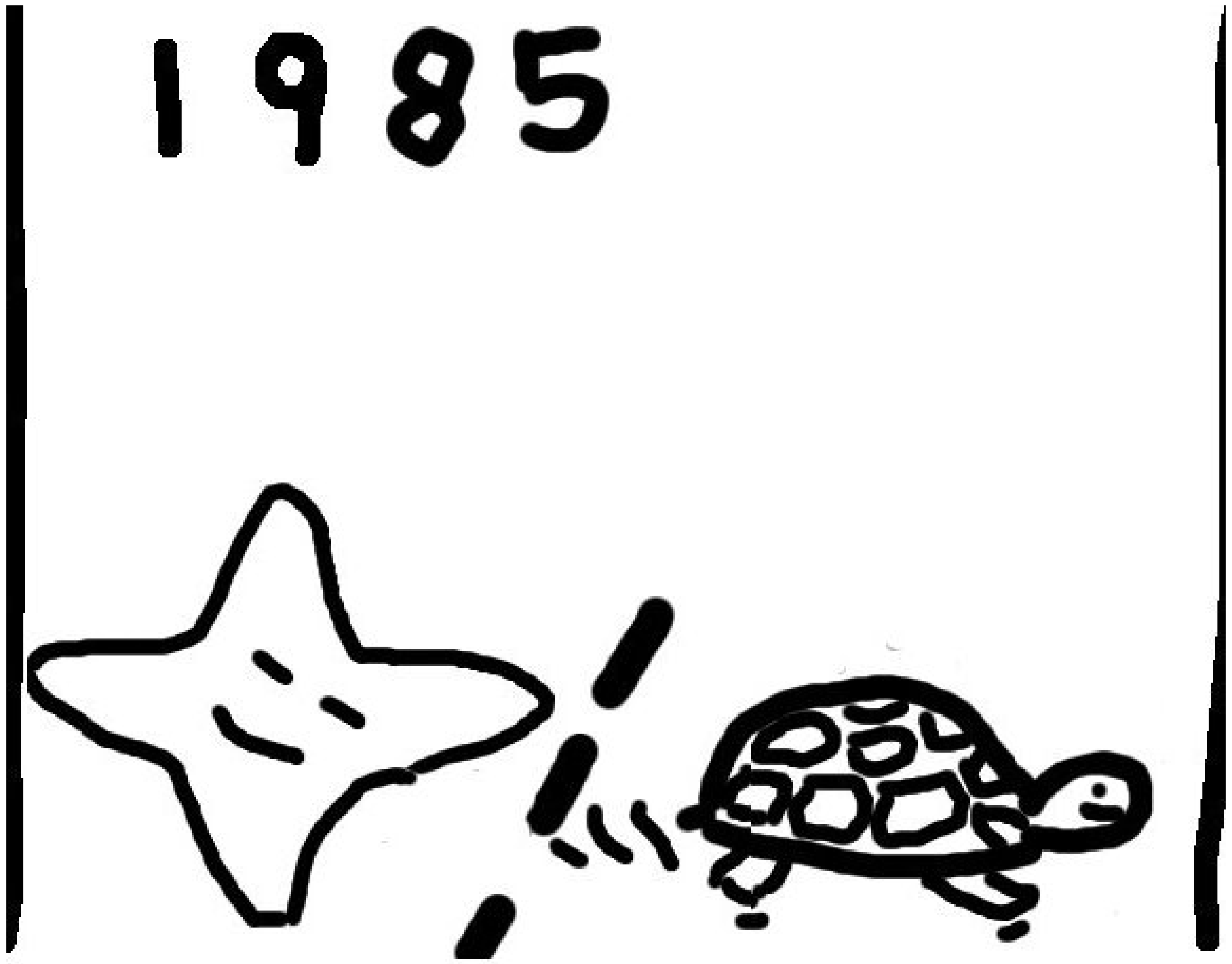


Weierstrass sets off, Edwards left behind sleeping

Weierstrass has made some progress
finally Edwards wakes up.



Weierstrass sets off, Edwards
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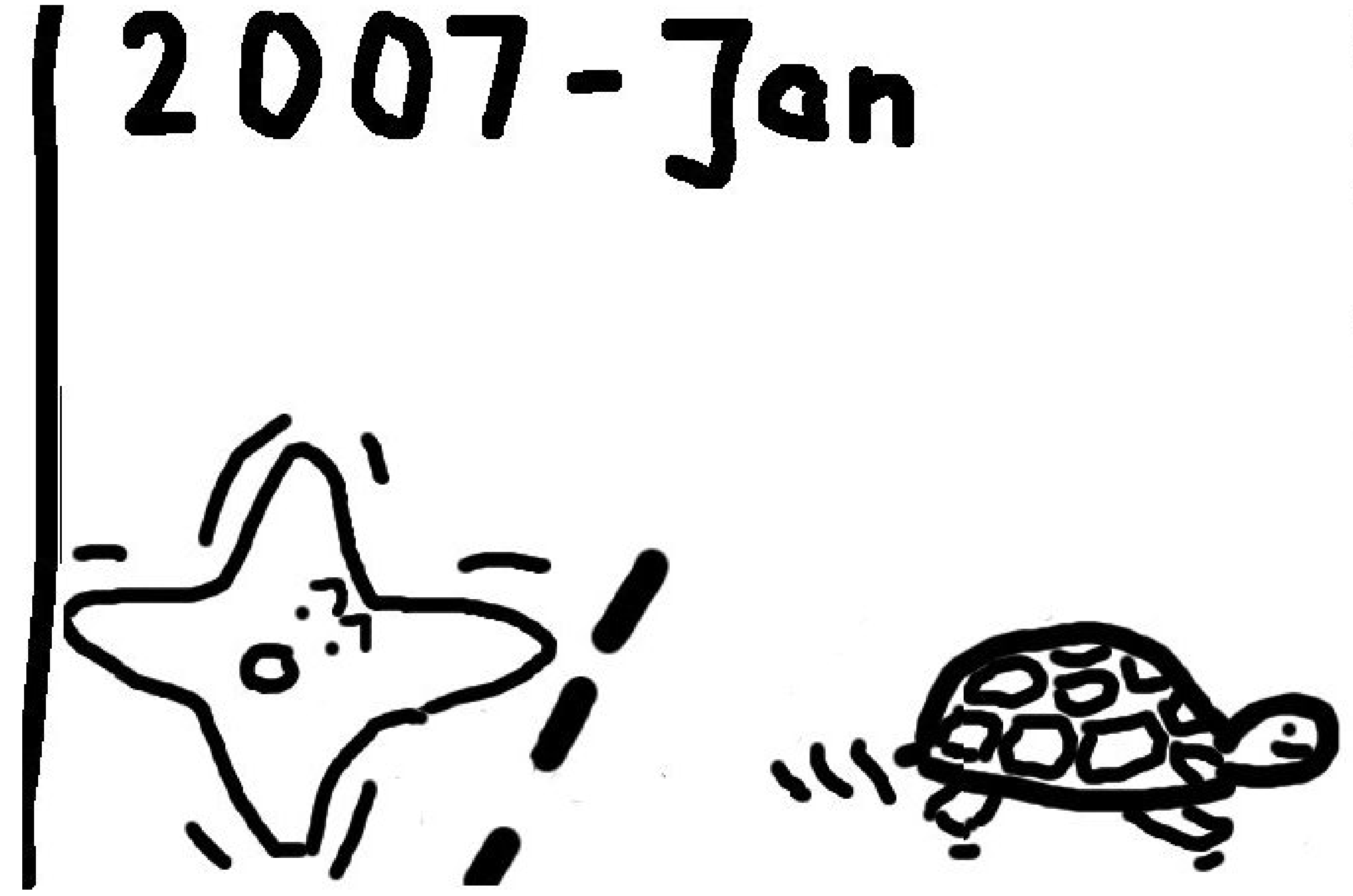

Weierstrass has made some progress –
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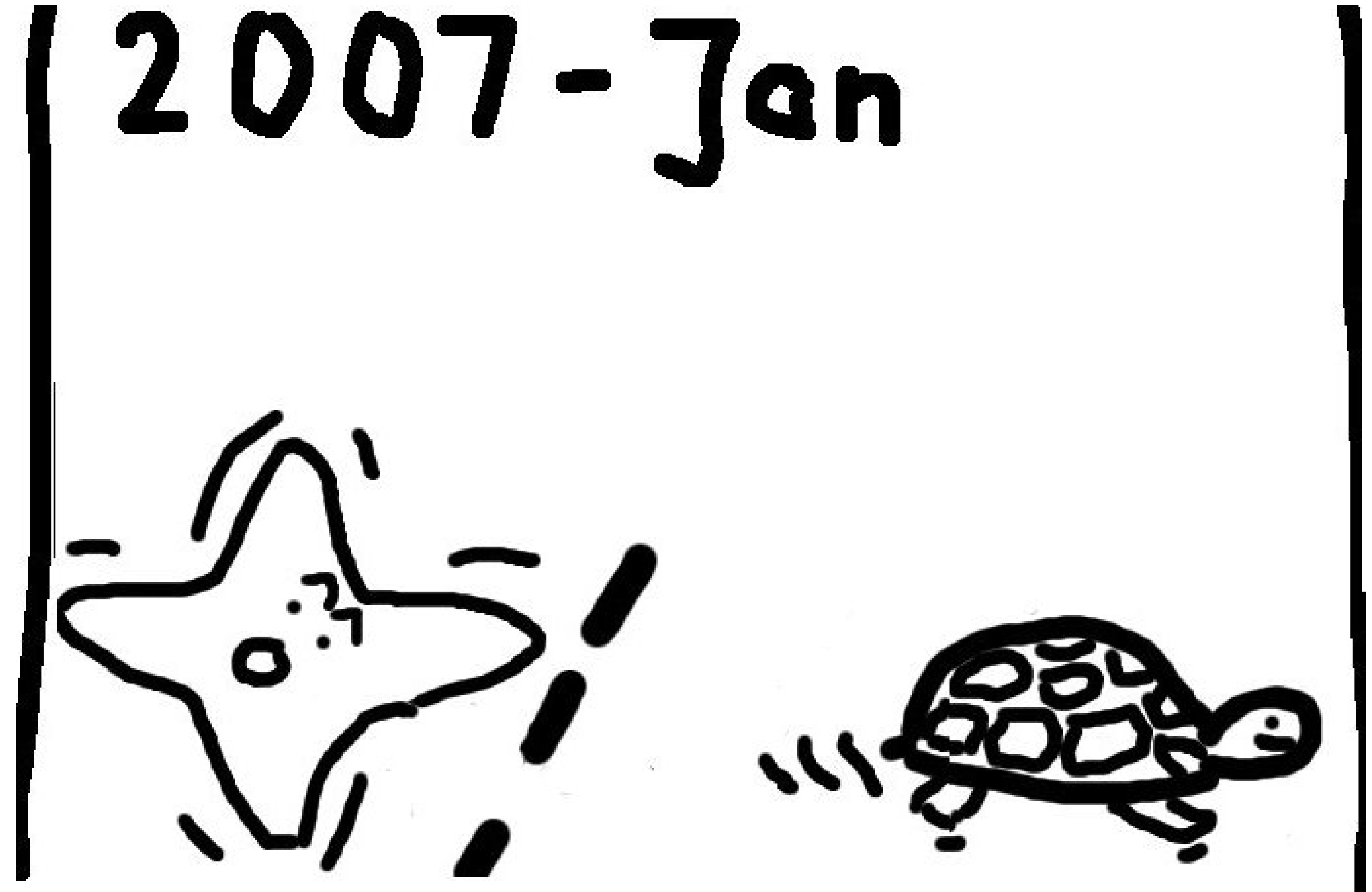

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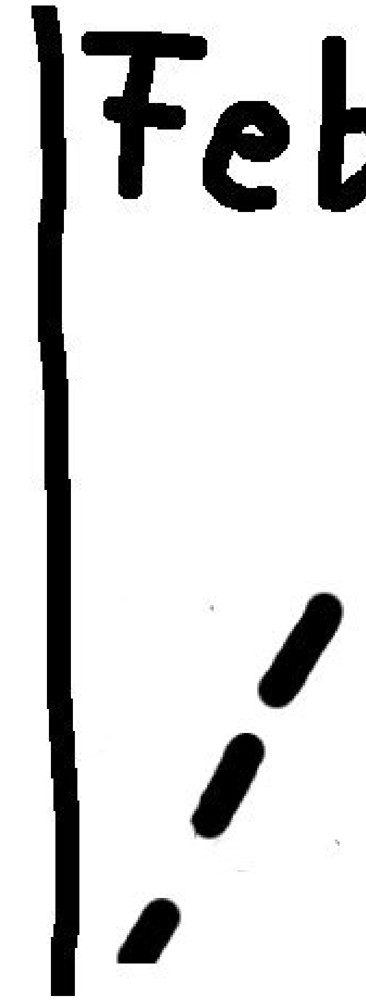

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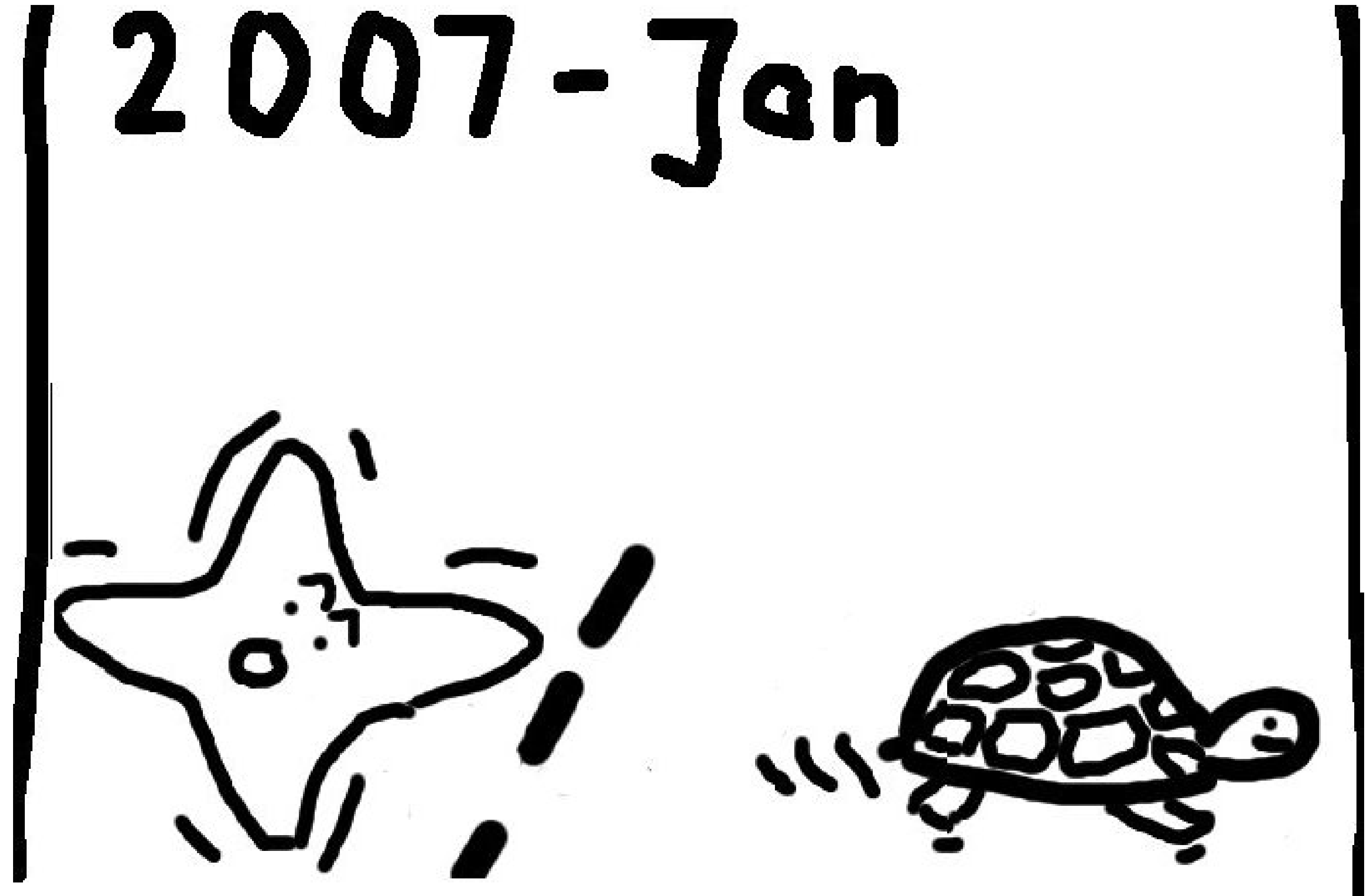

Weierstrass has made some progress -
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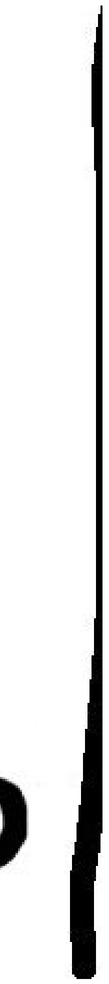
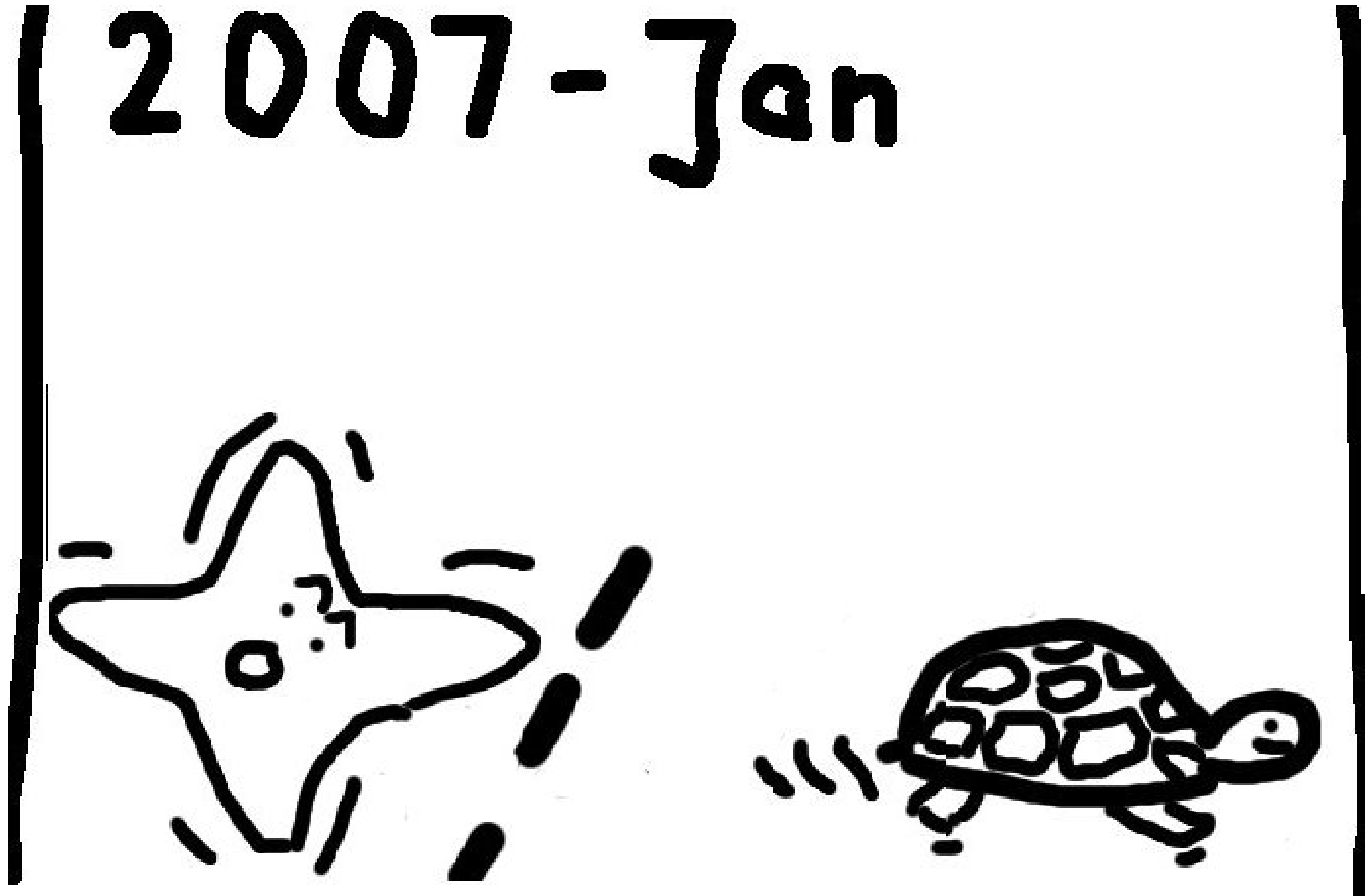
Exciting progress: Edwa
about to overtake!!





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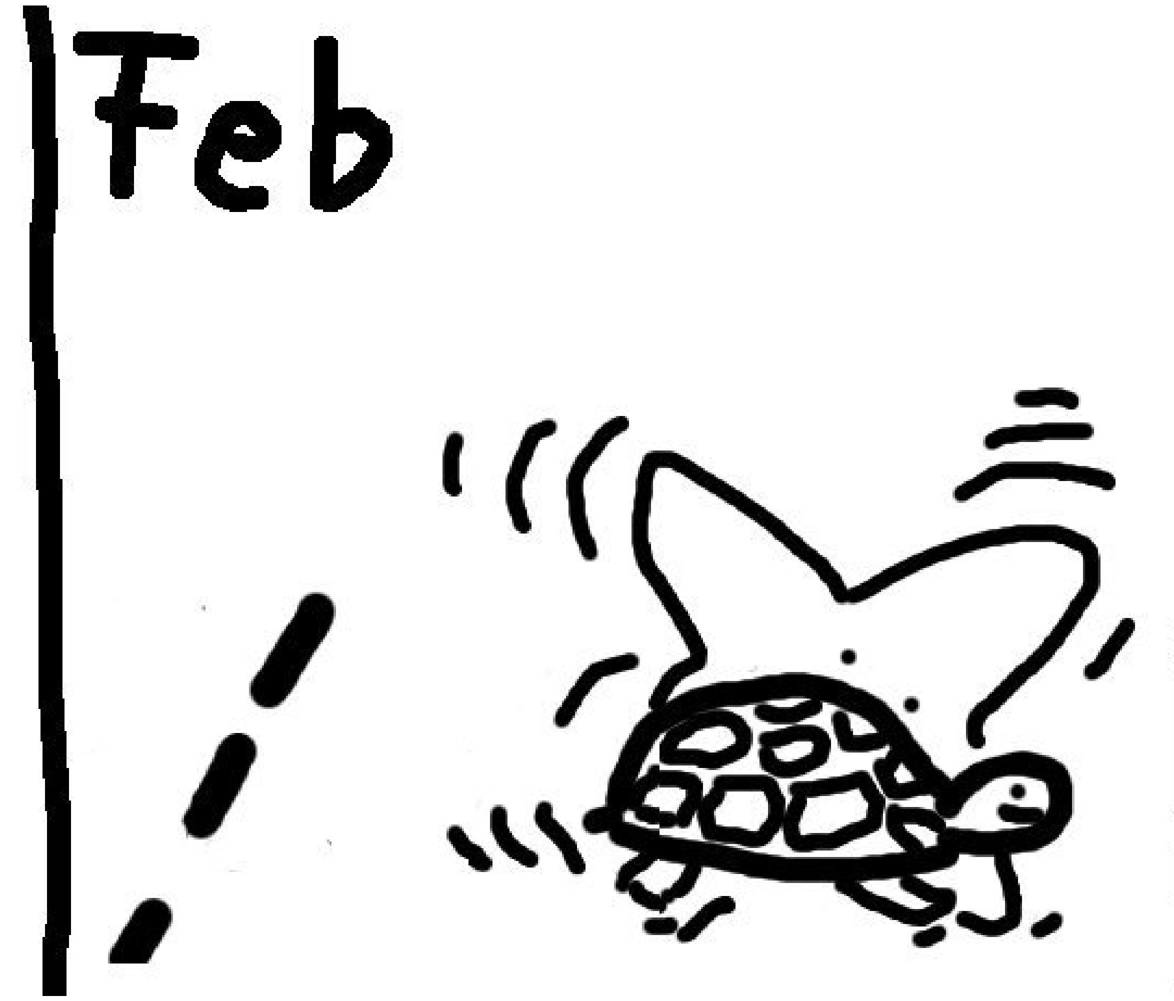

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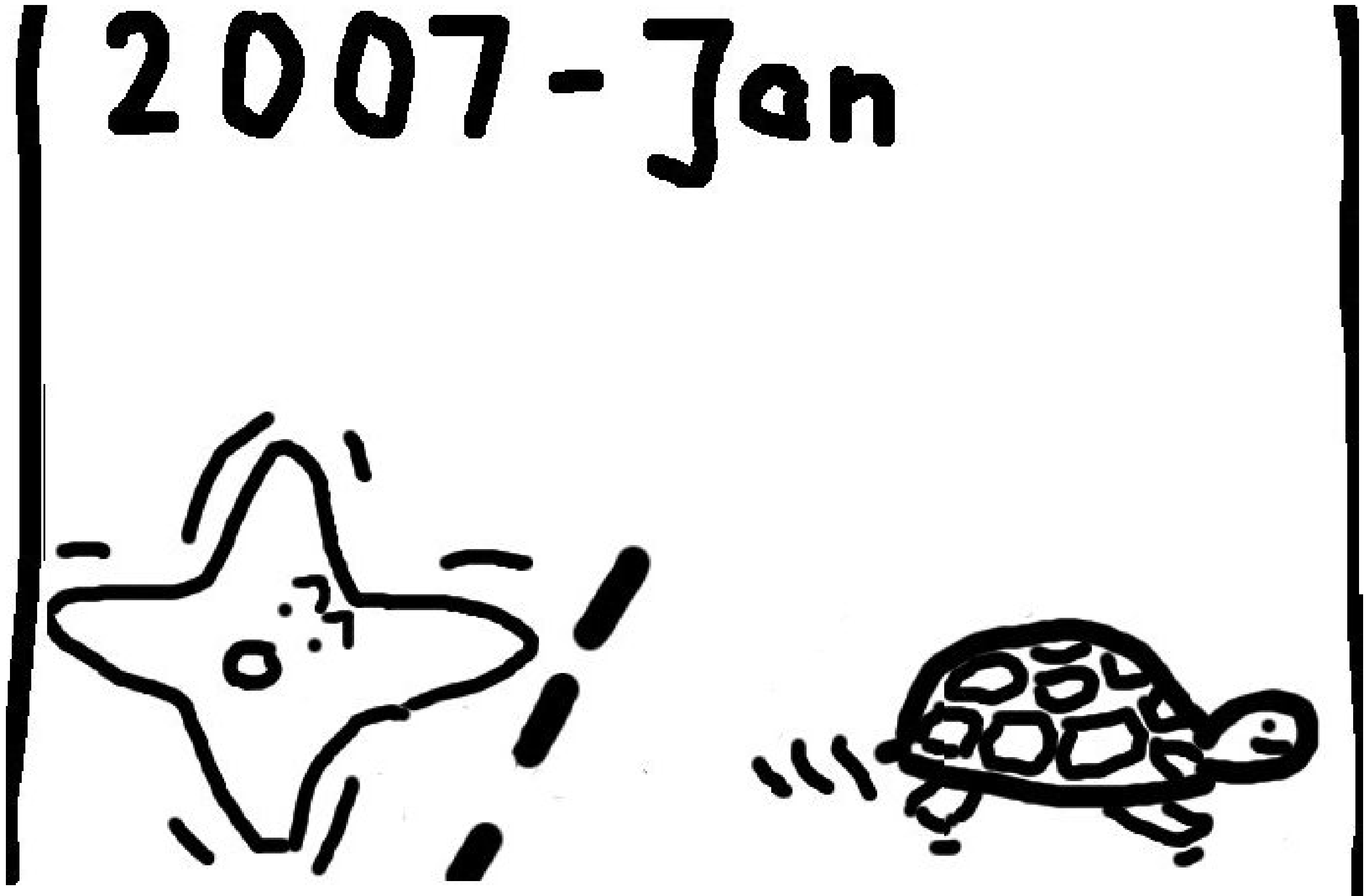

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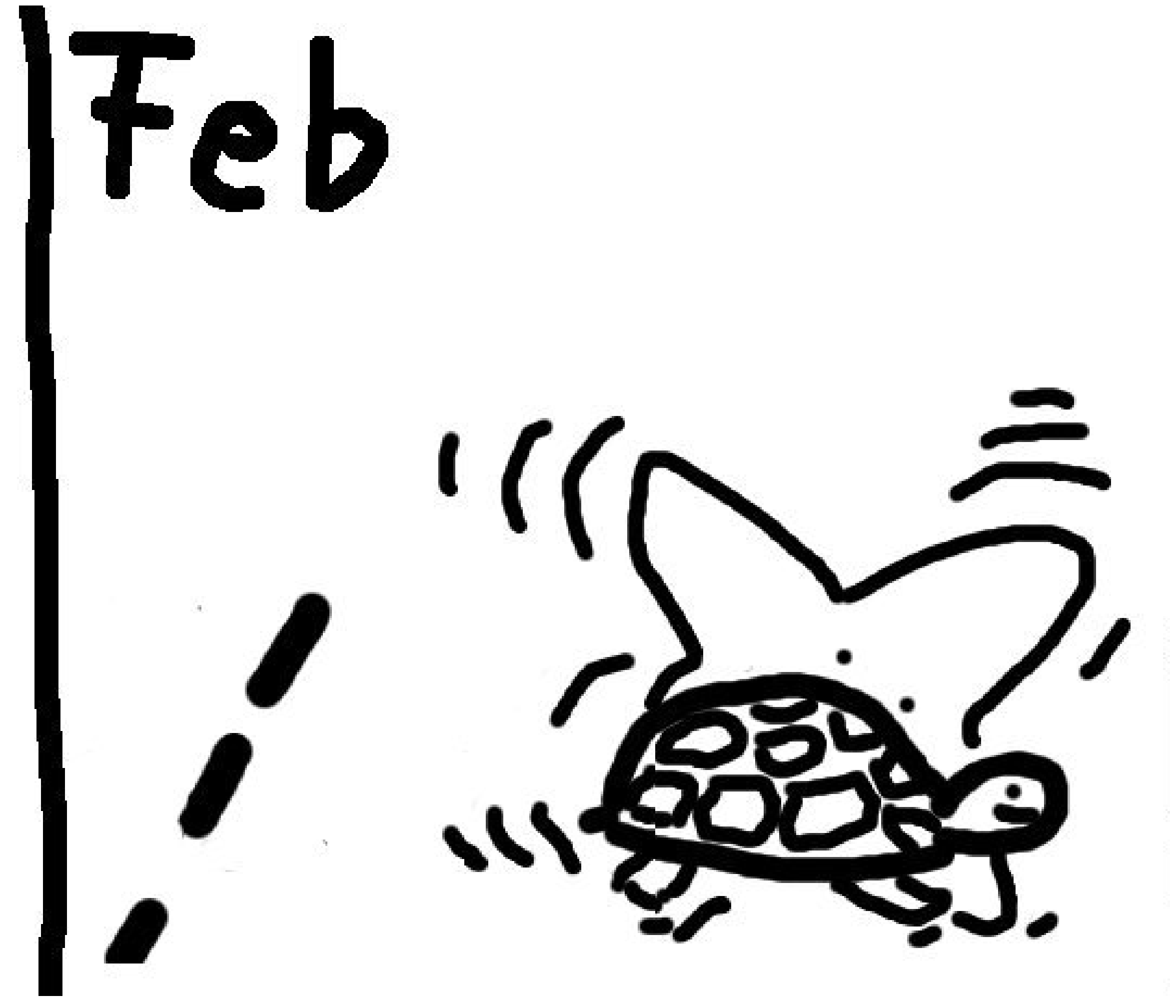

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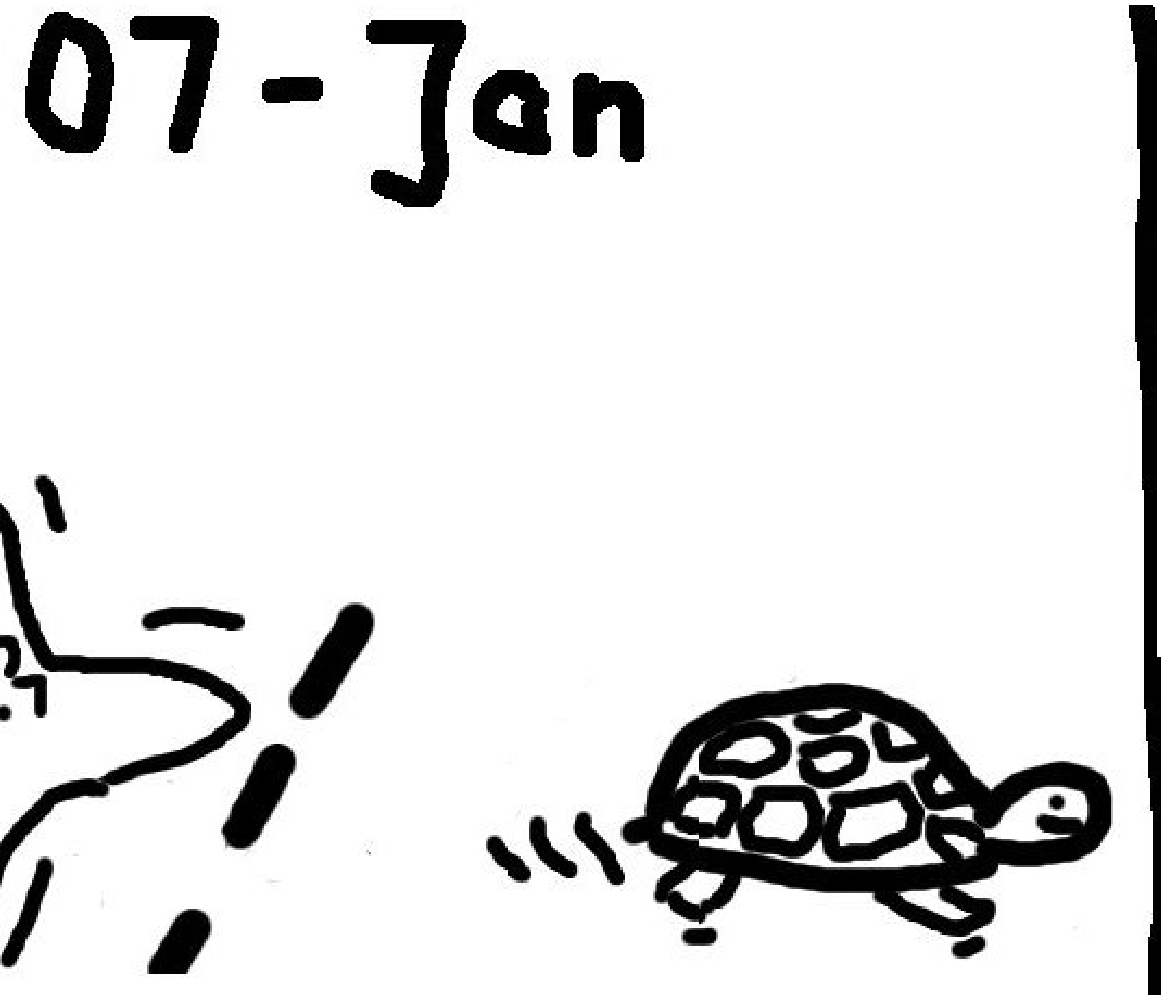

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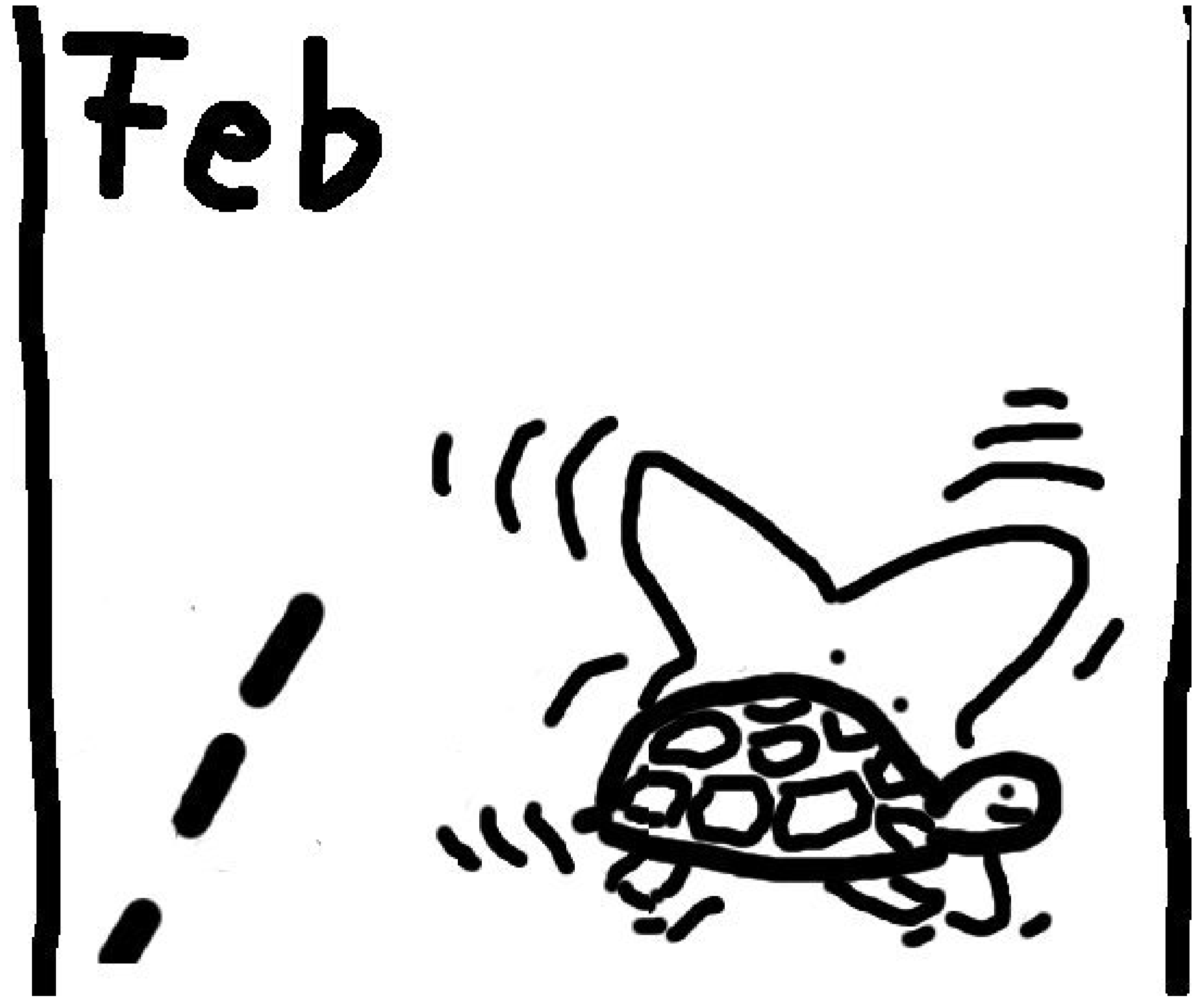
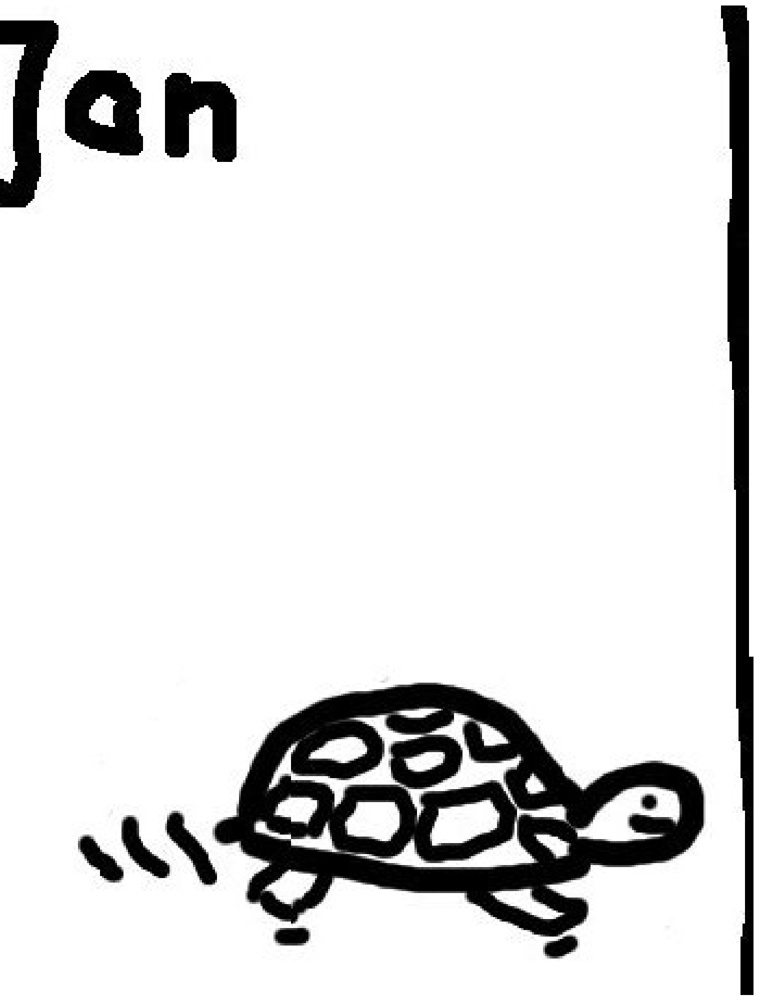


Exciting progress: Edwards
about to overtake!!

And the w



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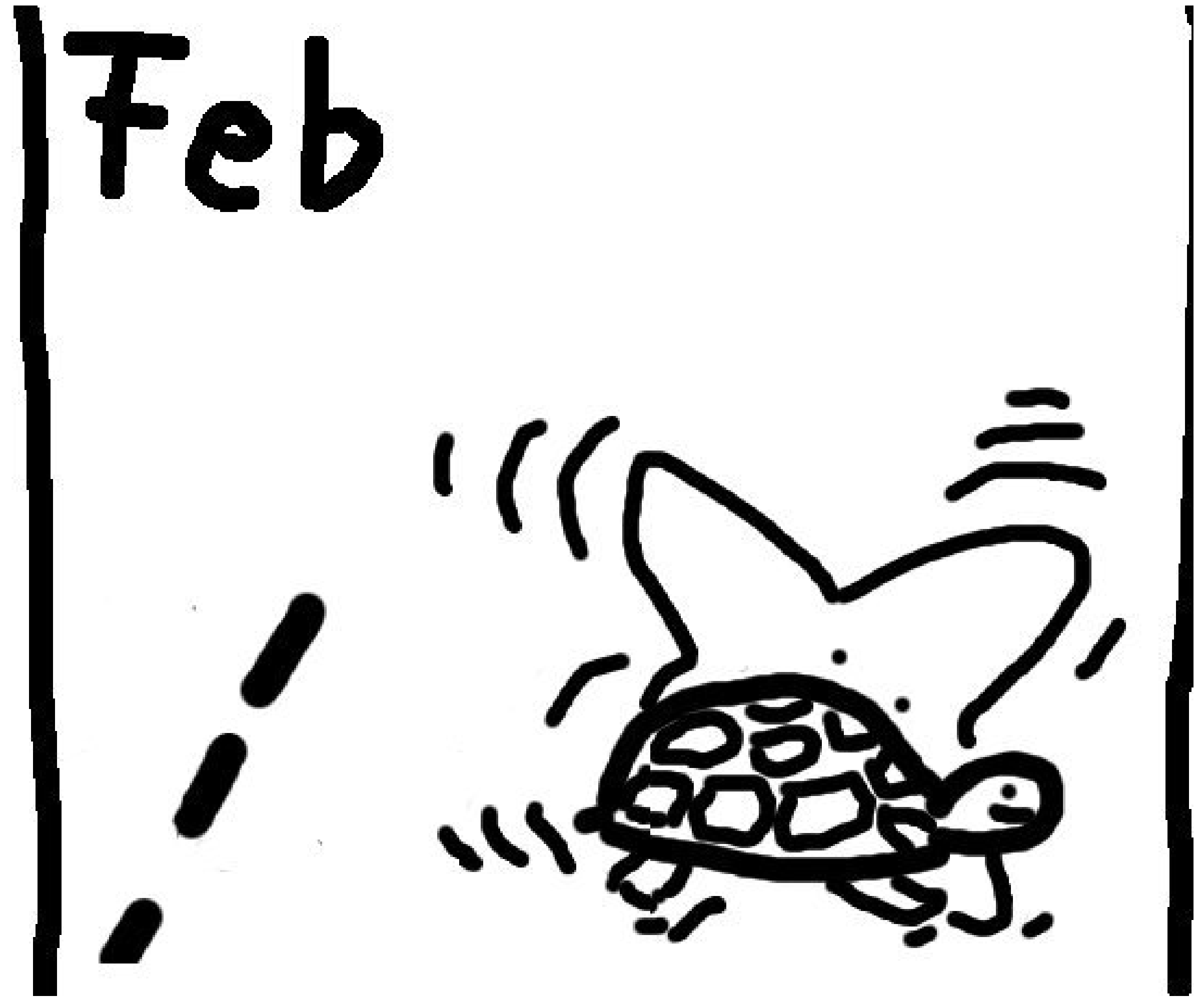
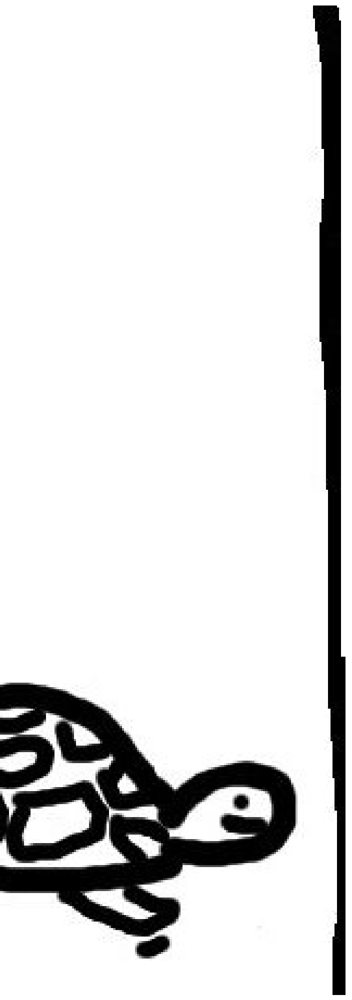
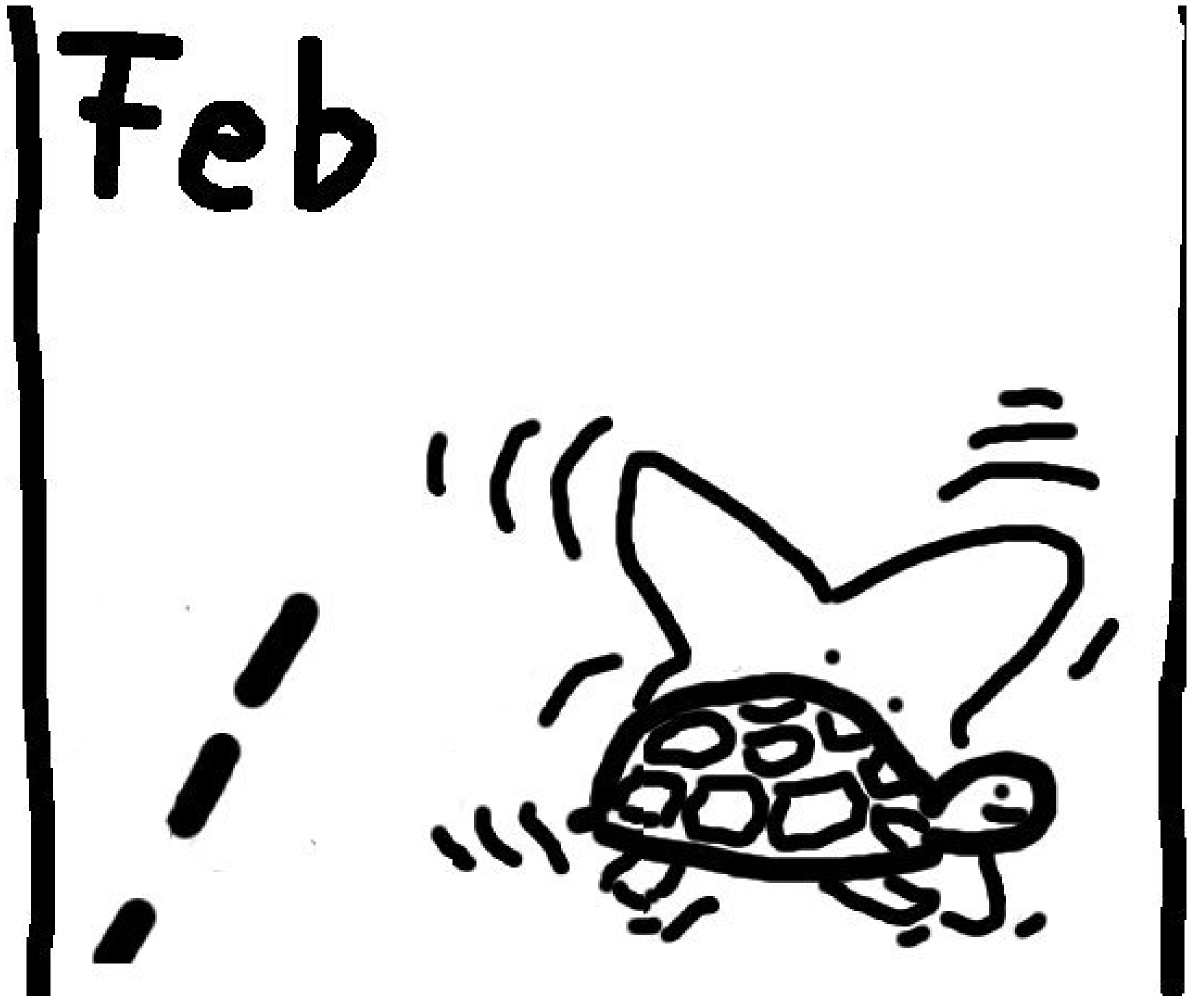


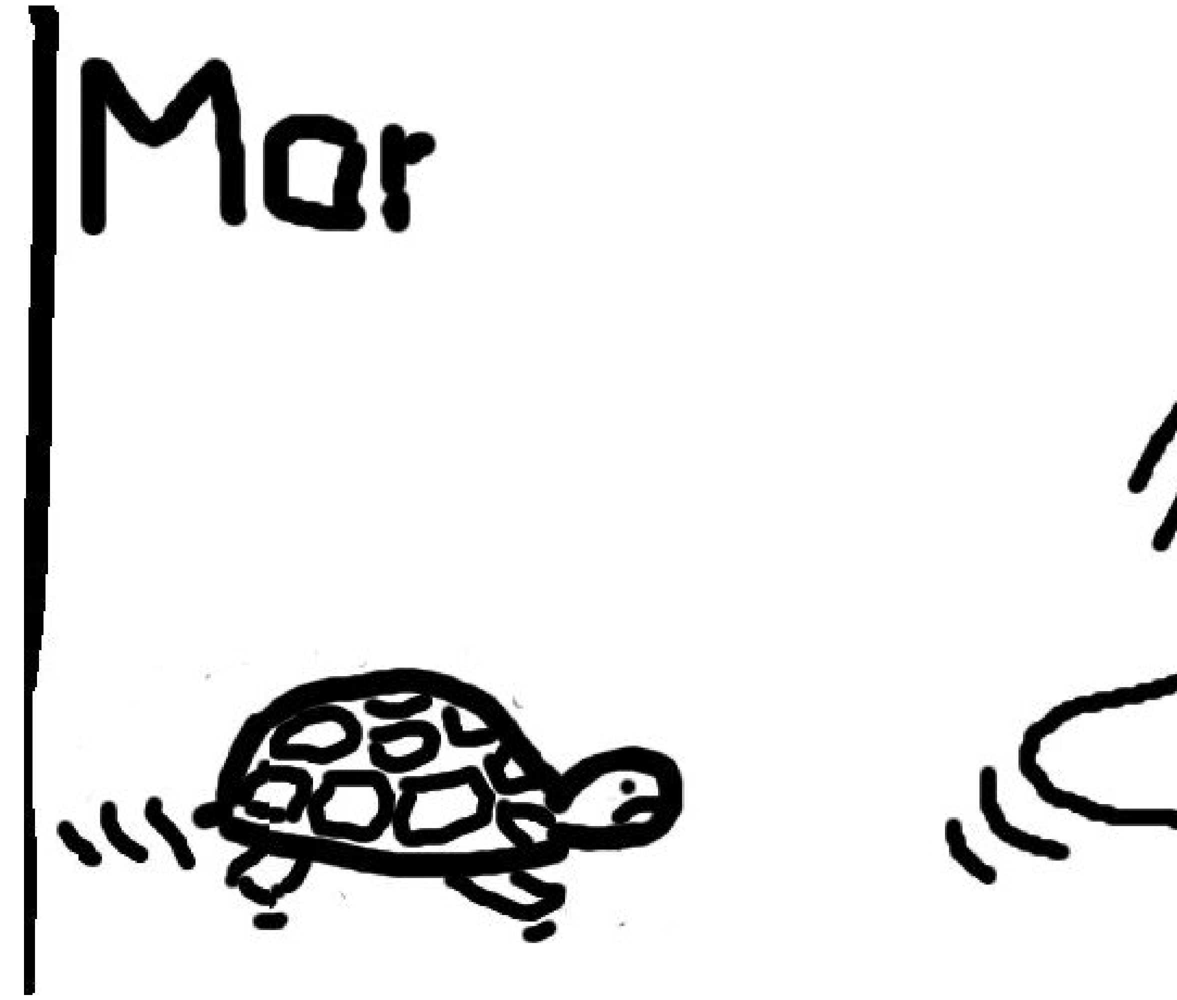
Exciting progress: Edwards about to overtake!!

And the winner is: Edw





Exciting progress: Edwards about to overtake!!



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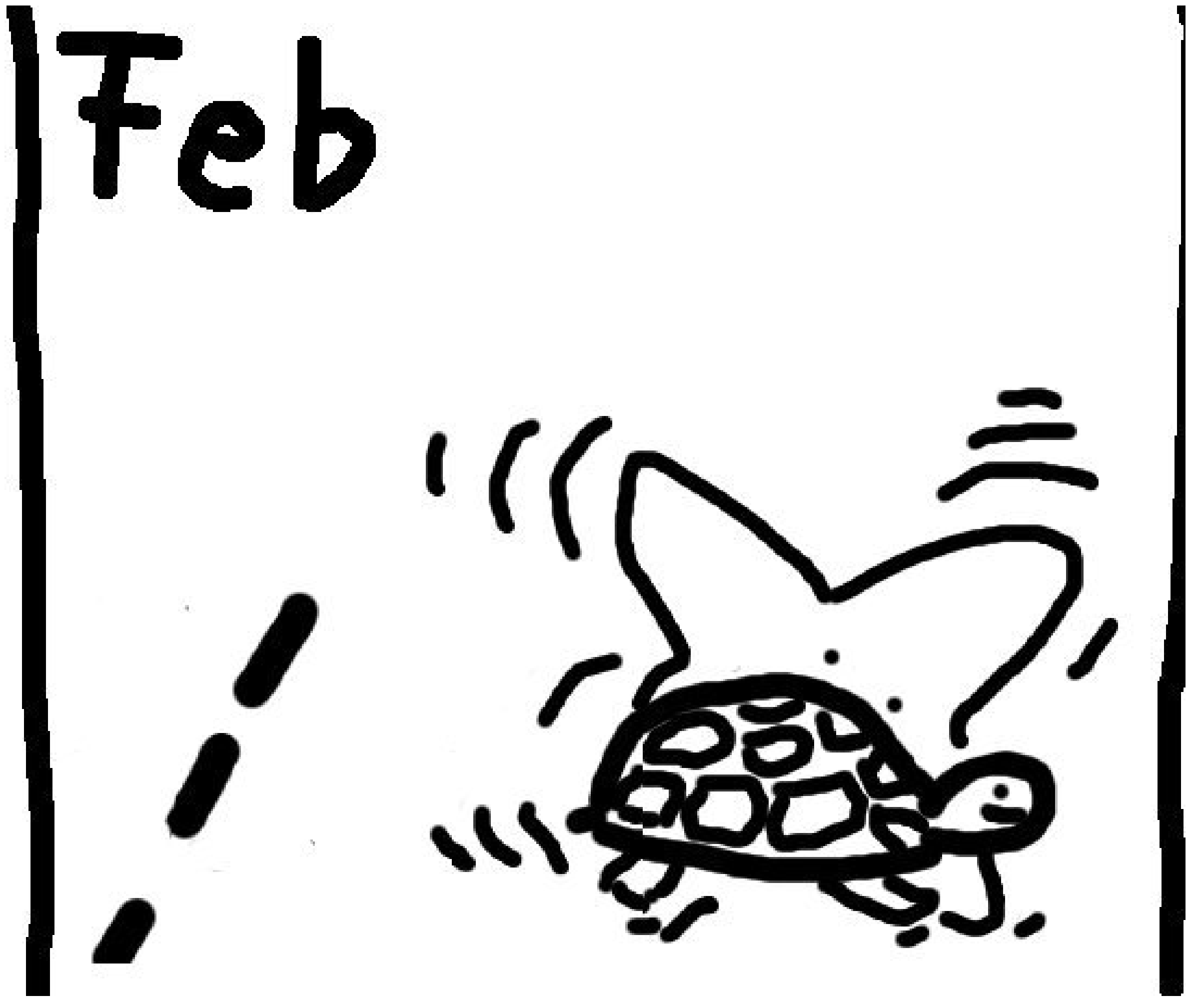

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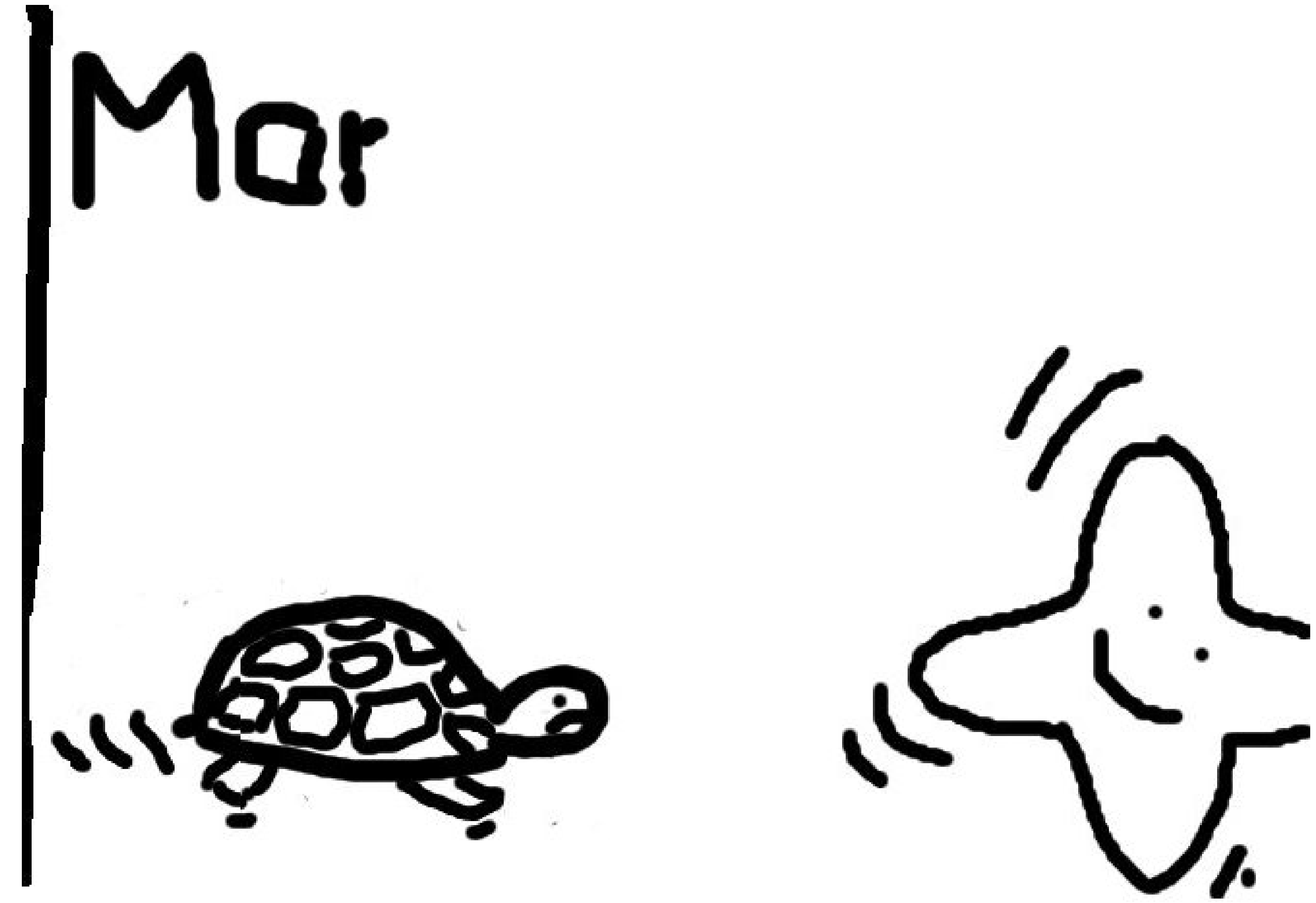

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