Complexity news: FFTs and integer multiplication

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Fix a field k with $2 \neq 0$.

Fix non-square $d \in k$.

$$\{(x, y) \in k \times k : \\ x^2 + y^2 = 1 + dx^2y^2\}$$

is a commutative group with

$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$

defined by Edwards addition law:

$$x_3 = \frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2}.$$

More on Edwards coordinates:

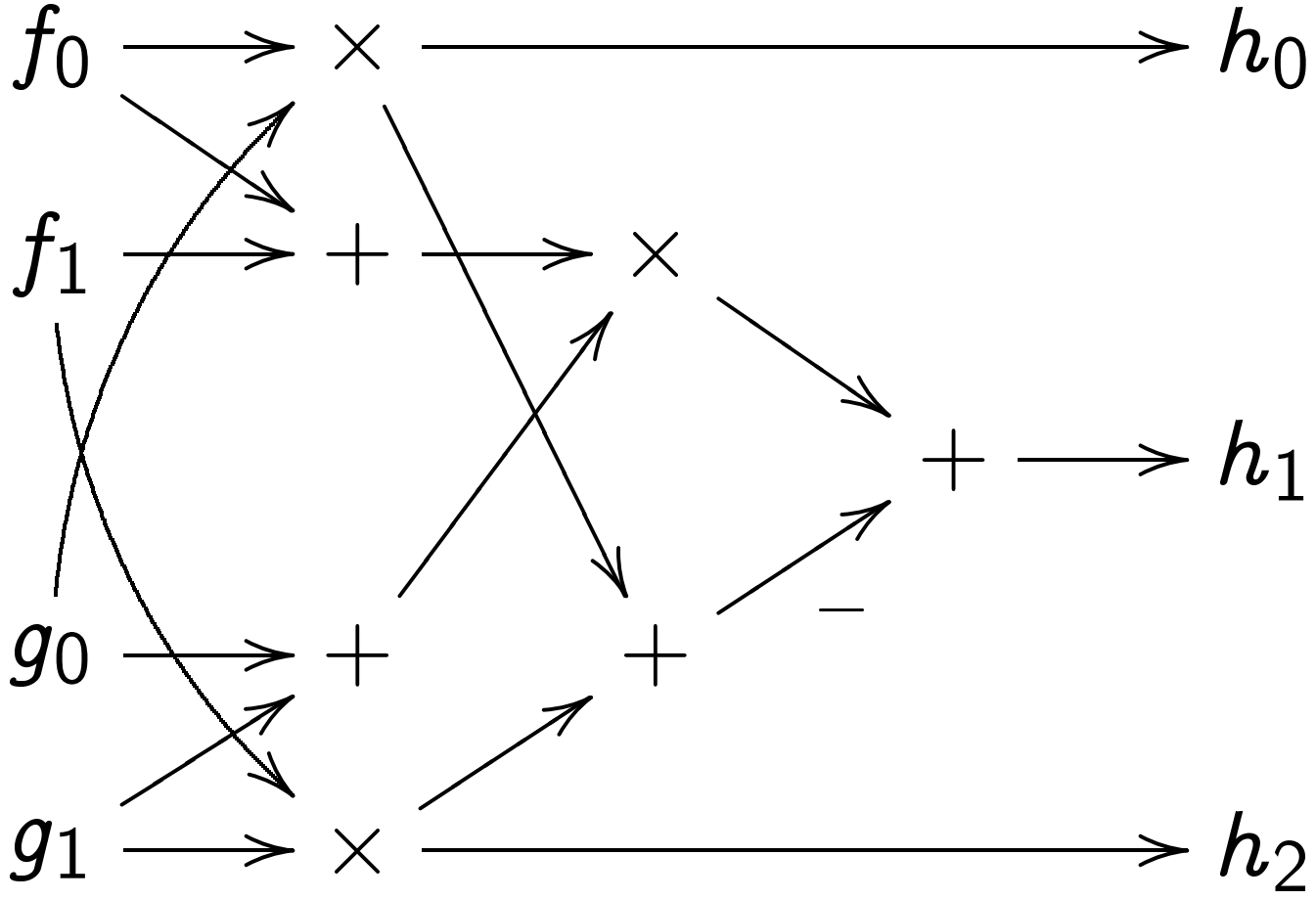
Harold M. Edwards,
"A normal form
for elliptic curves."

Daniel J. Bernstein, Tanja Lange,
"Faster addition and doubling
on elliptic curves."

`http://cr.yp.to`
`/newelliptic.html`

Algebraic complexity

Example (1963 Karatsuba):



This "**R**-algebraic algorithm" computes product $h_0 + h_1x + h_2x^2$ of $f_0 + f_1x, g_0 + g_1x \in \mathbf{R}[x]$.

"Total **R**-complexity" 7:

3 mults, 4 adds/subs.

FFT news

How quickly can we multiply
in the ring $\mathbf{R}[x]$?

1866 Gauss "FFT" $\Rightarrow$
algebraic algorithm to compute
$f, g \mapsto fg$ when $\deg f, \deg g < n$;
$\mathbf{R}$-complexity $(15 + o(1))n \lg n$.

1968 R. Yavne: Can do better!
"Split-radix FFT."
$\mathbf{R}$-complexity $(12 + o(1))n \lg n$.

2004 James Van Buskirk:
Can do better! "Tangent FFT."
$\mathbf{R}$-complexity $(34/3 + o(1))n \lg n$.

The FFT trick:

Unique $\mathbf{C}[x]$-algebra morphism
$\mathbf{C}[x]/(x^{1024}-1) \hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[x]/(x^{512}-1) \oplus \mathbf{C}[x]/(x^{512}+1)$.

Gauss continuation:
$\mathbf{C}[x]/(x^{512}+1) \hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[x]/(x^{256}-i) \oplus \mathbf{C}[x]/(x^{256}+i)$
where $i = \sqrt{-1}$; etc.

Alternative: Twist.

Unique $\mathbf{C}$-algebra morphism
$\mathbf{C}[x]/(x^{512}+1) \hookrightarrow\!\!\!\!\rightarrow \mathbf{C}[y]/(y^{512}-1)$
that maps x to $\zeta_{1024} y$
where $\zeta_{1024} = \exp(2\pi i/1024)$.
Same complexity as Gauss.

Split-radix FFT:
Don't twist $\mathbf{C}[x]/(x^{512}+1)$
but do twist $\mathbf{C}[x]/(x^{256} \pm i)$.

This maximizes the number of
multiplications by 4th roots of 1.

General principle:
"Align roots of 1."
Use $\mathbf{C}[x]/(x^{\cdots} - \omega)$
if ω is "aligned"
for easy multiplications.
Twist to avoid $\mathbf{C}[x]/(x^{\cdots} - \omega)$
if ω is not aligned.

The tangent FFT:

Rethink basis of $\mathbf{C}[x]/(x^n - 1)$.

Instead of $1, x, \ldots, x^{n-1}$ use

$1/s_{n,0}, x/s_{n,1}, \ldots, x^{n-1}/s_{n,n-1}$

where $s_{n,k} =$

$\max\left\{\left|\cos \frac{2\pi k}{n}\right|, \left|\sin \frac{2\pi k}{n}\right|\right\}$ ·

$\max\left\{\left|\cos \frac{2\pi k}{n/4}\right|, \left|\sin \frac{2\pi k}{n/4}\right|\right\}$ ·

$\max\left\{\left|\cos \frac{2\pi k}{n/16}\right|, \left|\sin \frac{2\pi k}{n/16}\right|\right\}$ ·

$\cdots$.

Note that $s_{n,k} = s_{n,k+n/4}$.

Note that $\zeta_n^k(s_{n/4,k}/s_{n,k})$ is

$\pm(1 + i \tan \cdots)$ or $\pm(\cot \cdots + i)$.

Complexity $8.5n - 28$ to split n

into $n/4, n/4, n/4, n/8, n/8$.

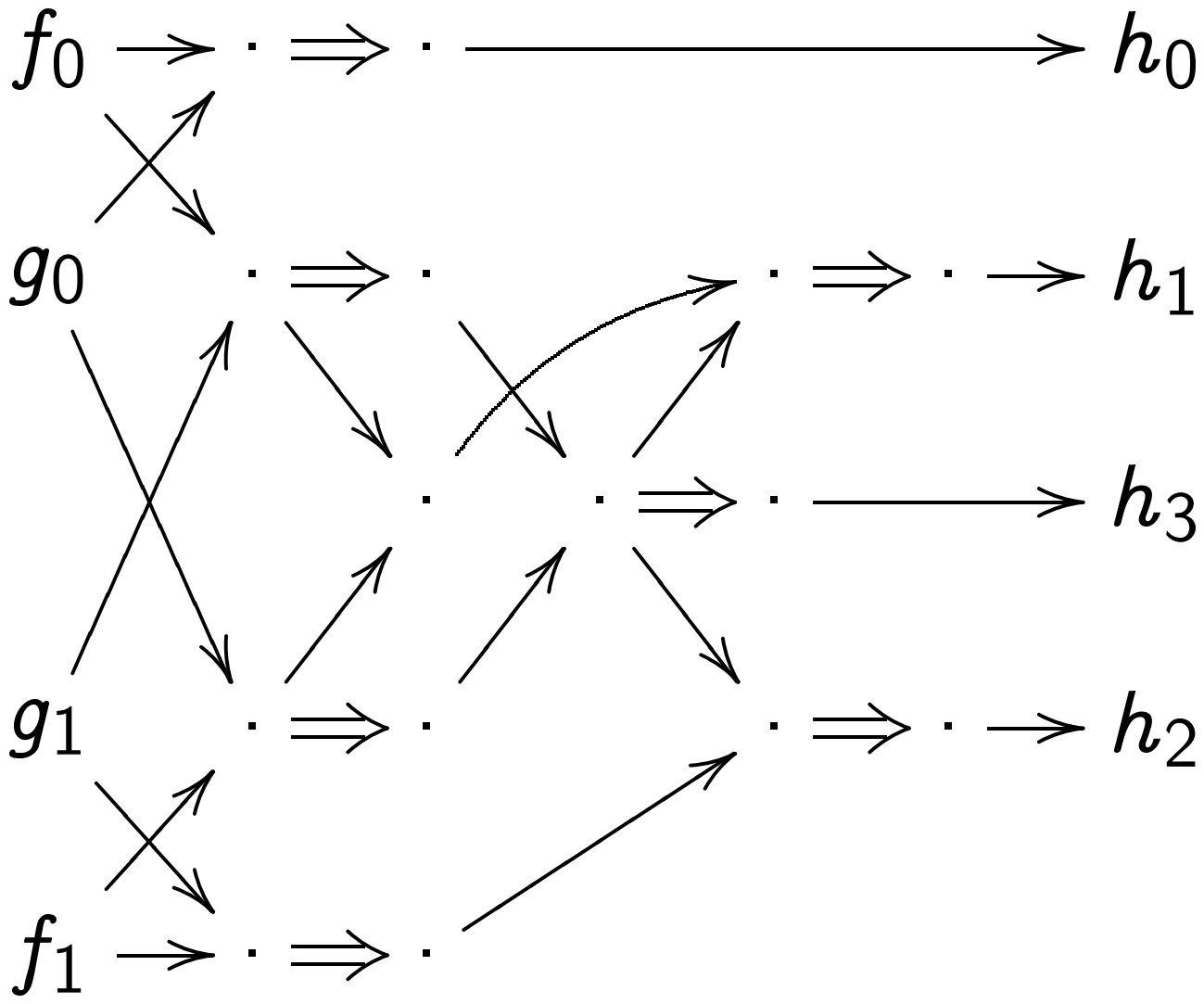
Three expositions of
how the tangent FFT works:

Frigo/Johnson,
in *IEEE Trans. Signal Processing*.

Lundy/Van Buskirk,
in *Computing*.

Bernstein, `http://cr.yp.to`
`/talks.html#2007.04.17`,
expanding an old idea of Fiduccia.

Bit complexity



Each $\cdot$ is a "gate" $a, b \mapsto 1 - ab$.

This "circuit" computes product $h_0 + 2h_1 + 4h_2 + 8h_3$ of $f_0 + 2f_1, g_0 + 2g_1 \in \mathbf{Z}$.

"Bit complexity" 15: 15 gates.

Integer-multiplication news

How quickly can we multiply in the ring $\mathbf{Z}$?

1971 Schönhage/Strassen: circuit to compute $f, g \mapsto fg$ when $f, g \in \{0, 1, \ldots, 2^n - 1\}$; bit complexity $\Theta(n \lg n \lg \lg n)$.

2007 Fürer: Can do better! Bit complexity $(n \lg n) 2^{O(\lg^* n)}$.

$\lg^* n = 1$ if $1 \leq \lg n < 2$;
$\lg^* n = 2$ if $1 \leq \lg \lg n < 2$;
$\lg^* n = 3$ if $1 \leq \lg \lg \lg n < 2$;
$\lg^* n = 4$ if $1 \leq \lg \lg \lg \lg n < 2$;
etc.

Bit complexity $n(\lg n)^{O(1)}$:
View n-bit integers as
$\Theta(n/\lg n)$-coeff polynomials
over a prime field k with
$\lg \#k \in \Theta(\lg n)$, appropriate ζ.
(Or $k = \mathbf{R}$, approximating coeffs.)

Bottleneck: FFT has many
mults by constants in k.

Schönhage/Strassen idea:
View n-bit integers as
$\Theta(\sqrt{n})$-coeff polynomials
over the ring $\mathbf{Z}/(2^{\Theta(\sqrt{n})} + 1)$.
Mults by $2^{\cdots}$ are easy in this ring.
$\Theta(\lg \lg n)$ levels of recursion.

Fürer idea:

View n-bit integers as
$\Theta(n/(\lg n)^2)$-coeff polynomials
over the ring $k[y]/(y^{\Theta(\lg n)} + 1)$
with $\lg \#k \in \Theta(\lg n)$,
appropriate ζ. (Or $k = \mathbf{R}$.)

$\Theta(n/\lg n)$ mults in ring.

Align roots in FFT.
(Recall split-radix speedup!)
Then most mults are easy,
as in Schönhage-Strassen.

$\Theta(n/(\lg n)^2)$ hard mults; ok.
$\Theta(\lg^* n)$ levels of recursion.