The tangent FFT

D. J. Bernstein
University of Illinois at Chicago

The convolution problem

How quickly can we multiply polynomials in the ring $\mathbf{R}[x]$?

Answer depends on degrees, representation of polynomials, number of polynomials, etc.

Answer also depends on definition of "quickly."
Many models of computation; many interesting cost measures.

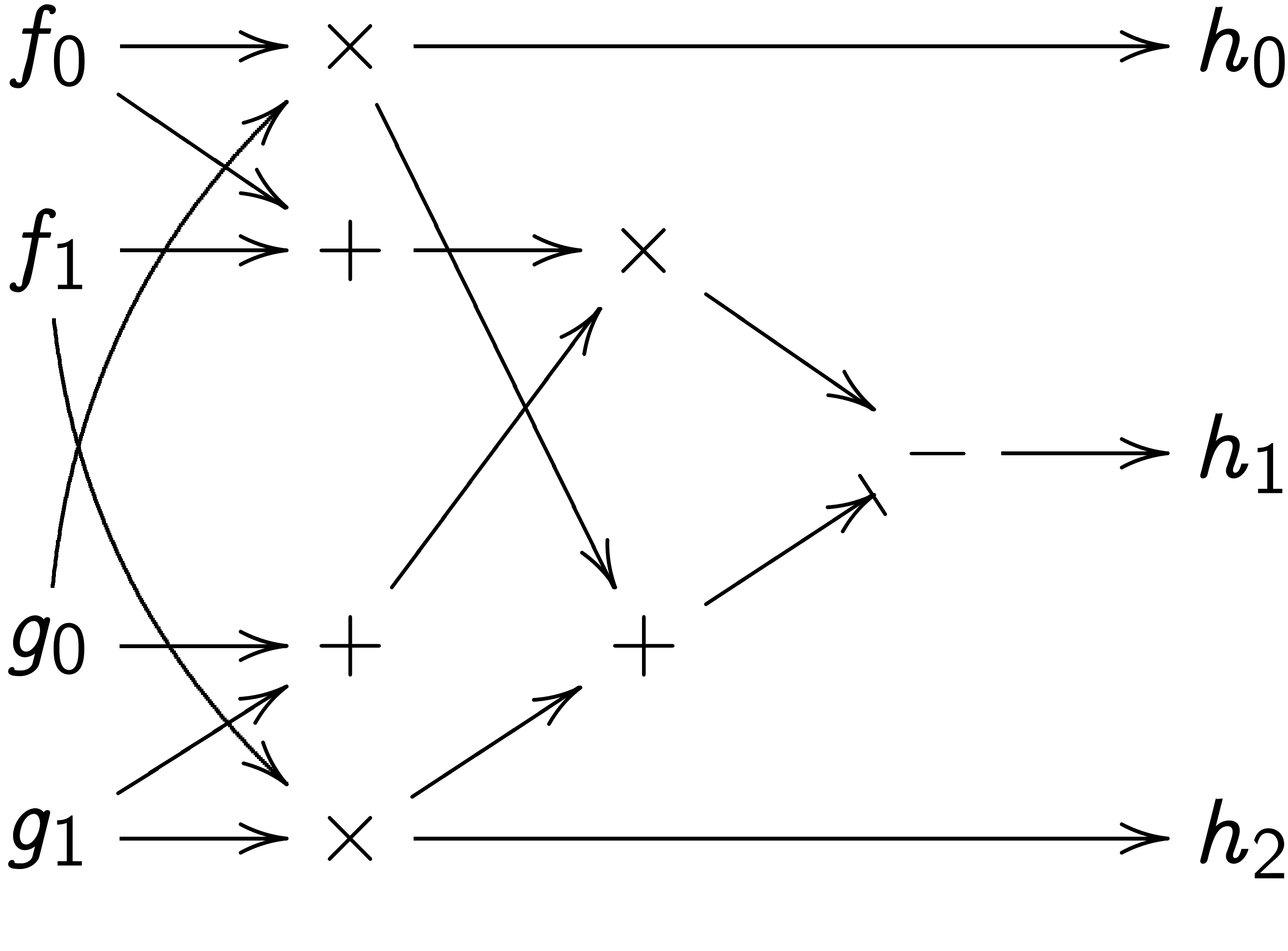
Assume two inputs $f, g \in \mathbf{R}[x]$, $\deg f < m$, $\deg g \le n - m$, so $\deg fg < n$. Assume f, g, fg represented as coeff sequences.

How quickly can we compute the n coeffs of fg, given f's m coeffs and g's $n - m + 1$ coeffs?

Inputs $(f_0, f_1, \ldots, f_{m-1}) \in \mathbf{R}^m$, $(g_0, g_1, \ldots, g_{n-m}) \in \mathbf{R}^{n-m+1}$.
Output $(h_0, h_1, \ldots, h_{n-1}) \in \mathbf{R}^n$ with $h_0 + h_1 x + \cdots + h_{n-1} x^{n-1} = (f_0 + f_1 x + \cdots)(g_0 + g_1 x + \cdots)$.

Assume **R**-algebraic algorithms (without divs, branches): chains of binary **R**-adds $u, v \mapsto u + v$, binary **R**-subs $u, v \mapsto u - v$, binary **R**-mults $u, v \mapsto uv$, starting from inputs and constants.

Example (1963 Karatsuba):



Cost measure for this talk:
total **R**-algebraic complexity.
Cost 1 for binary **R**-add;
cost 1 for binary **R**-sub;
cost 1 for binary **R**-mult;
cost 0 for constant in **R**.

Many real-world computations
use (e.g.) Pentium M's
floating-point operations
to approximate operations in **R**.
Properly scheduled operations
achieve Pentium M cycles
$\approx$ total **R**-algebraic complexity.

Fast Fourier transforms

Define $\zeta_n \in \mathbf{C}$ as $\exp(2\pi i/n)$.
Define $T : \mathbf{C}[x]/(x^n - 1) \hookrightarrow\!\!\twoheadrightarrow \mathbf{C}^n$
as $f \mapsto f(1), f(\zeta_n), \ldots, f(\zeta_n^{n-1})$.

Can very quickly compute T.

First publication of fast algorithm:
1866 Gauss.

Easy to see that Gauss's FFT uses
$O(n \lg n)$ arithmetic operations
if $n \in \{1, 2, 4, 8, \ldots\}$.

Several subsequent reinventions,
ending with 1965 Cooley Tukey.

Inverse map is also very fast.

Multiplication in $\mathbf{C}^n$ is very fast.

1966 Sande, 1966 Stockham:
Can very quickly multiply
in $\mathbf{C}[x]/(x^n - 1)$ or $\mathbf{C}[x]$ or $\mathbf{R}[x]$
by mapping $\mathbf{C}[x]/(x^n - 1)$ to $\mathbf{C}^n$.

Given $f, g \in \mathbf{C}[x]/(x^n - 1)$:
compute fg as $T^{-1}(T(f)T(g))$.

Given $f, g \in \mathbf{C}[x]$ with $\deg fg < n$:
compute fg from
its image in $\mathbf{C}[x]/(x^n - 1)$.

$\mathbf{R}$-algebraic complexity $O(n \lg n)$.

A closer look at costs

More precise analysis of Gauss FFT (and Cooley-Tukey FFT):

$\mathbf{C}[x]/(x^n - 1) \hookrightarrow\!\!\!\!\twoheadrightarrow \mathbf{C}^n$ using
$(1/2)n \lg n$ binary $\mathbf{C}$-adds,
$(1/2)n \lg n$ binary $\mathbf{C}$-subs,
$(1/2)n \lg n$ binary $\mathbf{C}$-mults,
if $n \in \{1, 2, 4, 8, \ldots\}$.

$(a, b) \in \mathbf{R}^2$ represents $a + bi \in \mathbf{C}$.
$\mathbf{C}$-add, $\mathbf{C}$-sub, $\mathbf{C}$-mult cost $2, 2, 6$:
$(a, b) + (c, d) = (a + c, b + d)$,
$(a, b) - (c, d) = (a - c, b - d)$,
$(a, b)(c, d) = (ac - bd, ad + bc)$.

Total cost $5n \lg n$.

Easily save some time:
eliminate mults by 1;
absorb mults by $-1, i, -i$
into subsequent operations;
simplify mults by $\pm\sqrt{\pm i}$
using, e.g., $(a, b)(1/\sqrt{2}, 1/\sqrt{2}) = ((a-b)/\sqrt{2}, (a+b)/\sqrt{2})$.

Cost $5n \lg n - 10n + 16$
to map $\mathbf{C}[x]/(x^n - 1) \hookrightarrow \mathbf{C}^n$,
if $n \in \{4, 8, 16, 32, \ldots\}$.

What about cost of convolution?

$5n \lg n + O(n)$ to compute $T(f)$,
$5n \lg n + O(n)$ to compute $T(g)$,
$O(n)$ to multiply in $\mathbf{C}^n$,
similar $5n \lg n + O(n)$ for T^{-1}.

Total cost $15n \lg n + O(n)$
to compute $fg \in \mathbf{C}[x]/(x^n - 1)$
given $f, g \in \mathbf{C}[x]/(x^n - 1)$.

Total cost $(15/2)n \lg n + O(n)$
to compute $fg \in \mathbf{R}[x]/(x^n - 1)$
given $f, g \in \mathbf{R}[x]/(x^n - 1)$: map
$\mathbf{R}[x]/(x^n - 1) \hookrightarrow\!\!\!\!\rightarrow \mathbf{R}^2 \oplus \mathbf{C}^{n/2-1}$
(Gauss) to save half the time.

1968 R. Yavne: Can do better!

Cost $4n \lg n - 6n + 8$

to map $\mathbf{C}[x]/(x^n - 1) \hookrightarrow\!\!\!\!\rightarrow \mathbf{C}^n$,

if $n \in \{2, 4, 8, 16, \ldots\}$.

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if $n \in \{2, 4, 8, 16, \ldots\}$.

2004 James Van Buskirk:
Can do better!
Cost $(34/9)n \lg n + O(n)$.

Expositions of the new algorithm:
Frigo, Johnson,
in *IEEE Trans. Signal Processing*;
Lundy, Van Buskirk,
in *Computing*;
Bernstein, this talk,
expanding an old idea of Fiduccia.

Van Buskirk, `comp.arch`, January 2005: "Have you ever considered changing djbfft to get better opcounts along the lines of `home.comcast.net/~kmbtib`?"

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Van Buskirk, `comp.arch`: "Oh, you're so 20th century, Dan. Read the handwriting on the wall."

Understanding the FFT

The FFT trick: $\mathbf{C}[x]/(x^n - 1) \hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[x]/(x^{n/2} - 1) \oplus \mathbf{C}[x]/(x^{n/2} + 1)$
by unique $\mathbf{C}[x]$-algebra morphism.
Cost $2n$: $n/2$ $\mathbf{C}$-adds, $n/2$ $\mathbf{C}$-subs.

e.g. $n = 4$: $\mathbf{C}[x]/(x^4 - 1) \hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[x]/(x^2 - 1) \oplus \mathbf{C}[x]/(x^2 + 1)$
by $g_0 + g_1 x + g_2 x^2 + g_3 x^3 \mapsto$
$(g_0 + g_2) + (g_1 + g_3)x$,
$(g_0 - g_2) + (g_1 - g_3)x$.
Representation: $(g_0, g_1, g_2, g_3) \mapsto$
$((g_0 + g_2), (g_1 + g_3))$,
$((g_0 - g_2), (g_1 - g_3))$.

Recurse: $\mathbf{C}[x]/(x^{n/2} - 1) \hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[x]/(x^{n/4} - 1) \oplus \mathbf{C}[x]/(x^{n/4} + 1)$;
similarly $\mathbf{C}[x]/(x^{n/2} + 1) \hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[x]/(x^{n/4} - i) \oplus \mathbf{C}[x]/(x^{n/4} + i)$;
continue to $\mathbf{C}[x]/(x - 1) \oplus \cdots$.

General case: $\mathbf{C}[x]/(x^n - \alpha^2) \hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[x]/(x^{n/2} - \alpha) \oplus \mathbf{C}[x]/(x^{n/2} + \alpha)$
by $g_0 + \cdots + g_{n/2}x^{n/2} + \cdots \mapsto$
$(g_0 + \alpha g_{n/2}) + (g_1 + \alpha \cdots)x + \cdots$,
$(g_0 - \alpha g_{n/2}) + (g_1 - \alpha \cdots)x + \cdots$.
Cost $5n$: $n/2$ $\mathbf{C}$-mults,
$n/2$ $\mathbf{C}$-adds, $n/2$ $\mathbf{C}$-subs.

Recurse, eliminate easy mults.
Cost $5n \lg n - 10n + 16$.

Alternative: the twisted FFT

After previous $\mathbf{C}[x]/(x^n - 1) \hookrightarrow\!\!\!\!\rightarrow \mathbf{C}[x]/(x^{n/2}-1) \oplus \mathbf{C}[x]/(x^{n/2}+1)$, apply unique $\mathbf{C}$-algebra morphism $\mathbf{C}[x]/(x^{n/2}+1) \hookrightarrow\!\!\!\!\rightarrow \mathbf{C}[y]/(y^{n/2}-1)$ that maps x to $\zeta_n y$.

$g_0 + g_1 x + \cdots + g_{n/2} x^{n/2} + \cdots \mapsto (g_0 + g_{n/2}) + (g_1 + g_{n/2+1})x + \cdots, (g_0 - g_{n/2}) + \zeta_n (g_1 - g_{n/2+1}) y + \cdots.$
Again cost $5n$: $n/2$ $\mathbf{C}$-mults, $n/2$ $\mathbf{C}$-adds, $n/2$ $\mathbf{C}$-subs.

Eliminate easy mults, recurse.
Cost $5n \lg n - 10n + 16$.

The split-radix FFT

FFT and twisted FFT end up with same number of mults by ζ_n, same number of mults by $\zeta_{n/2}$, same number of mults by $\zeta_{n/4}$, etc.

Is this necessary? No! Split-radix FFT: more easy mults. "Don't twist until you see the whites of their i's."

(Can use same idea to speed up Schönhage-Strassen algorithm for integer multiplication.)

Cost $2n$: $\mathbf{C}[x]/(x^n - 1) \hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[x]/(x^{n/2} - 1) \oplus \mathbf{C}[x]/(x^{n/2} + 1)$.
Cost n: $\mathbf{C}[x]/(x^{n/2} + 1) \hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[x]/(x^{n/4} - i) \oplus \mathbf{C}[x]/(x^{n/4} + i)$.
Cost $6(n/4)$: $\mathbf{C}[x]/(x^{n/4} - i) \hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[y]/(y^{n/4} - 1)$ by $x \mapsto \zeta_n y$.
Cost $6(n/4)$: $\mathbf{C}[x]/(x^{n/4} + i) \hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[y]/(y^{n/4} - 1)$ by $x \mapsto \zeta_n^{-1} y$.

Overall cost $6n$ to split into
$1/2, 1/4, 1/4$, entropy 1.5 bits.

Eliminate easy mults, recurse.
Cost $4n \lg n - 6n + 8$,
exactly as in 1968 Yavne.

The tangent FFT

Several ways to achieve
cost 6 for mult by $e^{i\theta}$.

One approach: Factor $e^{i\theta}$
as $(1 + i \tan\theta)\cos\theta$.
Cost 2 for mult by $\cos\theta$.
Cost 4 for mult by $1 + i\tan\theta$.

For stability and symmetry,
use $\max\{|\cos\theta|, |\sin\theta|\}$
instead of $\cos\theta$.

Surprise (Van Buskirk):
Can merge some cost-2 mults!

Rethink basis of $\mathbf{C}[x]/(x^n - 1)$.
Instead of $1, x, \ldots, x^{n-1}$ use
$1/s_{n,0}, x/s_{n,1}, \ldots, x^{n-1}/s_{n,n-1}$
where $s_{n,k} =$
$\max\left\{\left|\cos \frac{2\pi k}{n}\right|, \left|\sin \frac{2\pi k}{n}\right|\right\} \cdot$
$\max\left\{\left|\cos \frac{2\pi k}{n/4}\right|, \left|\sin \frac{2\pi k}{n/4}\right|\right\} \cdot$
$\max\left\{\left|\cos \frac{2\pi k}{n/16}\right|, \left|\sin \frac{2\pi k}{n/16}\right|\right\} \cdot$
$\cdots$.

Now $(g_0, g_1, \ldots, g_{n-1})$ represents
$g_0/s_{n,0} + \cdots + g_{n-1}x^{n-1}/s_{n,n-1}$.

Note that $s_{n,k} = s_{n,k+n/4}$.
Note that $\zeta_n^k(s_{n/4,k}/s_{n,k})$ is
$\pm(1 + i \tan \cdots)$ or $\pm(\cot \cdots + i)$.

Cost $2n$:

$\mathbf{C}[x]/(x^n - 1)$, basis $x^k/s_{n,k}$, $\hookrightarrow\!\!\!\!\twoheadrightarrow$

$\mathbf{C}[x]/(x^{n/2} - 1)$, basis $x^k/s_{n,k}$, $\oplus$

$\mathbf{C}[x]/(x^{n/2} + 1)$, basis $x^k/s_{n,k}$.

Cost n:

$\mathbf{C}[x]/(x^{n/2} - 1)$, basis $x^k/s_{n,k}$, $\hookrightarrow\!\!\!\!\twoheadrightarrow$

$\mathbf{C}[x]/(x^{n/4} - 1)$, basis $x^k/s_{n,k}$, $\oplus$

$\mathbf{C}[x]/(x^{n/4} + 1)$, basis $x^k/s_{n,k}$.

Cost n:

$\mathbf{C}[x]/(x^{n/2} + 1)$, basis $x^k/s_{n,k}$, $\hookrightarrow\!\!\!\!\twoheadrightarrow$

$\mathbf{C}[x]/(x^{n/4} - i)$, basis $x^k/s_{n,k}$, $\oplus$

$\mathbf{C}[x]/(x^{n/4} + i)$, basis $x^k/s_{n,k}$.

Cost $n/2-2$:

$\mathbf{C}[x]/(x^{n/4}-1)$, basis $x^k/s_{n,k}$, $\hookrightarrow\!\!\!\!\twoheadrightarrow$
$\mathbf{C}[x]/(x^{n/4}-1)$, basis $x^k/s_{n/4,k}$.

Cost $n/2-2$:

$\mathbf{C}[x]/(x^{n/4}+1)$, basis $x^k/s_{n,k}$, $\hookrightarrow\!\!\!\!\twoheadrightarrow$
$\mathbf{C}[x]/(x^{n/4}+1)$, basis $x^k/s_{n/2,k}$.

Cost $n/2$:

$\mathbf{C}[x]/(x^{n/4}+1)$, basis $x^k/s_{n/2,k}$,

$\hookrightarrow\!\!\!\!\twoheadrightarrow$

$\mathbf{C}[x]/(x^{n/8}-i)$, basis $x^k/s_{n/2,k}$,

$\oplus$

$\mathbf{C}[x]/(x^{n/8}+i)$, basis $x^k/s_{n/2,k}$.

Cost $4(n/4) - 6$:
$\mathbf{C}[x]/(x^{n/4} - i)$, basis $x^k/s_{n,k}$, $\hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[x]/(y^{n/4} - 1)$, basis $y^k/s_{n/4,k}$.

Cost $4(n/4) - 6$:
$\mathbf{C}[x]/(x^{n/4} + i)$, basis $x^k/s_{n,k}$, $\hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[x]/(y^{n/4} - 1)$, basis $y^k/s_{n/4,k}$.

Cost $4(n/8) - 6$:
$\mathbf{C}[x]/(x^{n/8} - i)$, basis $x^k/s_{n/2,k}$,
$\hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[x]/(y^{n/8} - 1)$, basis $y^k/s_{n/8,k}$.

Cost $4(n/8) - 6$:
$\mathbf{C}[x]/(x^{n/8} + i)$, basis $x^k/s_{n/2,k}$,
$\hookrightarrow\!\!\!\!\rightarrow$
$\mathbf{C}[x]/(y^{n/8} - 1)$, basis $y^k/s_{n/8,k}$.

Overall cost $8.5n - 28$ to split into $1/4, 1/4, 1/4, 1/8, 1/8$, entropy $9/4$.
Recurse: $(34/9)n \lg n + O(n)$.

What if input is in $\mathbf{C}[x]/(x^n - 1)$ with usual basis $1, x, \ldots, x^{n-1}$?
Could scale immediately, but faster to scale upon twist.
Cost $(34/9)n \lg n - (124/27)n - 2 \lg n - (2/9)(-1)^{\lg n} \lg n + (16/27)(-1)^{\lg n} + 8$,
exactly as in 2004 Van Buskirk.

Easily handle $\mathbf{R}[x]/(x^n+1)$
by mapping to $\mathbf{C}[x]/(x^{n/2}-i)$.

Easily handle $\mathbf{R}[x]/(x^n-1)$
by mapping to
$\mathbf{R}[x]/(x^{n/2}-1) \oplus \mathbf{R}[x]/(x^{n/2}+1)$.

Cost $(17/9)n \lg n + O(n)$ for
$\mathbf{R}[x]/(x^n-1) \hookrightarrow\!\!\!\!\twoheadrightarrow \mathbf{R}^2 \times \mathbf{C}^{n/2-1}$,
so cost $(17/3)n \lg n + O(n)$
to compute $fg \in \mathbf{R}[x]/(x^n-1)$
given $f, g \in \mathbf{R}[x]/(x^n-1)$.

Cost $(17/3)n \lg n + O(n)$
for size-n convolution.

Open: Can 17/3 be improved?