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Some New Algorithms for High-Precision Computation of Euler's Constant

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Abstract. We describe several new algorithms for the high-precision computation of Euler's constant $\gamma = 0.577\ldots$. Using one of the algorithms, which is based on an identity involving Bessel functions, γ has been computed to 30,100 decimal places. By computing their regular continued fractions we show that, if γ or $\exp(\gamma)$ is of the form P/Q for integers P and Q, then $|Q| > 10^{15000}$.

1. Introduction. Euler's constant γ is defined by

$$\gamma = \lim_{m\to\infty} (H_m - \ln(m)), \tag{1}$$

where $H_m = \Sigma_{k=1}^{m} 1/k$.

Recently $\zeta(3)$ was proved irrational [17] with the aid of a rapidly converging continued fraction, and conceivably a similar method might be used to prove the irrationality of γ. Thus, there is some interest in finding rapidly converging expressions for γ. We give several such expressions below.

Early computations of γ used the Euler-Maclaurin expansion to accelerate convergence of (1): see Brent [7] and Glaisher [12]. Sweeney [16] suggested a method which avoids the need for computation of the Bernoulli numbers which appear in the Euler-Maclaurin expansion, and Brent [7], [8] used Sweeney's method to compute γ to 20,700 decimal places. In Section 3 we describe an algorithm which is about twice as fast as Sweeney's. The algorithm depends on some identities, given in Section 2, involving modified Bessel functions. To demonstrate the effectiveness of the algorithm we have used it to compute γ to 30,100 decimal places; see Section 5. Some other algorithms for the high-precision computation of γ are briefly described and compared in Section 4.

2. Some Bessel Function Identities. The modified Bessel functions $I_\nu(z)$ and $K_0(z)$ are defined by

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)} \quad \text{and} \quad K_0(z) = -\partial I_\nu(z)/\partial\nu|_{\nu=0}.$$

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It is easy to verify, by carrying out the differentiation indicated above, that

$$\gamma + \ln(z/2) = \frac{S_0(z) - K_0(z)}{I_0(z)}, \tag{2}$$

where

$$S_0(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{(k!)^2} H_k.$$

For real positive z, $K_0(z)$ and $I_0(z)$ have the asymptotic expansions

$$K_0(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \sum_{k=0}^{\infty} (-1)^k a_k(z) \tag{3}$$

and

$$I_0(z) \sim (2\pi z)^{-1/2} e^{z} \sum_{k=0}^{\infty} a_k(z), \tag{4}$$

where

$$a_k(z) = \frac{1^2 3^2 \cdots (2k-1)^2}{k!(8z)^k} = \frac{[(2k)!]^2}{(k!)^3 (32z)^k}.$$

A proof is given in Watson [19, Section 7.23].

For real $z \geqslant 1$, the first terms in the asymptotic expansions (3) and (4) give upper and lower bounds on $K_0(z)$ and $I_0(z)$, respectively:

$$0 < K_0(z) < \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \quad \text{and} \quad I_0(z) > (2\pi z)^{-1/2} e^{z}.$$

Thus, taking $z = 2n \geqslant 2$ in (2), we have

$$0 < U(n)/V(n) - \gamma = K_0(2n)/I_0(2n) < \pi e^{-4n}, \tag{5}$$

where

$$U(n) = S_0(2n) - I_0(2n)\ln(n) = \sum_{k=0}^{\infty} \left(\frac{n^k}{k!}\right)^2 (H_k - \ln(n)) \tag{6}$$

and

$$V(n) = I_0(2n) = \sum_{k=0}^{\infty} \left(\frac{n^k}{k!}\right)^2. \tag{7}$$

In the following section we describe an algorithm (B1) for computing γ using (5) to (7). It is interesting to note that the relations (2) to (4) were essentially given by Riemann [14] in 1855, but the possibility of using them to compute Euler's constant appears to have been overlooked.

3. The Algorithm B1. Suppose we wish to evaluate γ to d decimal places. If we choose

$$n = \lfloor c + \tfrac{1}{4}\ln(10)d \rfloor \tag{8}$$

for some suitable constant c then, from (5),

$$|\gamma - U(n)/V(n)| < \pi e^{4-4c} 10^{-d}.$$

Thus, we need only evaluate $U(n)$ and $V(n)$ sufficiently accurately, and then perform one high-precision division, to evaluate γ to the required accuracy.

Let

$$A_k = \left(\frac{n^k}{k!}\right)^2 (H_k - \ln(n)), \qquad U_k = \sum_{j=0}^{k} A_j,$$

$$B_k = \left(\frac{n^k}{k!}\right)^2, \qquad V_k = \sum_{j=0}^{k} B_j.$$

Then

$$A_0 = -\ln(n), \quad B_0 = 1, \quad U_0 = A_0, \quad V_0 = 1, \tag{9}$$

and for $k = 1, 2, \ldots$, we have

$$\begin{aligned} B_k &= B_{k-1} n^2/k^2, \qquad A_k = (A_{k-1} n^2/k + B_k)/k, \\ U_k &= U_{k-1} + A_k, \qquad V_k = V_{k-1} + B_k. \end{aligned} \tag{10}$$

For Algorithm B1, n is chosen according to (8), and working precision equivalent to slightly more than d (floating) decimal places is used. $\ln(n)$ is computed, e.g. by the $O(d^2)$ method of [9], and A_0, B_0, U_0 and V_0 are initialized as in (9). The iteration (10) is terminated when, to the working precision, $U_k = U_{k-1}$ and $V_k = V_{k-1}$. The storage required is $O(d)$ as B_k can overwrite B_{k-1}, etc.

For $j \geqslant 0$, let α_j be the real positive root of

$$\alpha_j \ln \alpha_j - \alpha_j = j. \tag{11}$$

Thus, $\alpha_0 = e \simeq 2.78$, $\alpha_1 \simeq 3.59$, $\alpha_2 \simeq 4.32$, $\alpha_3 \simeq 4.97$, etc.

Using Stirling's approximation, we see that the number of iterations of (10) required is

$$K = \alpha_1 n + O(\ln(n)) = \tfrac{1}{4}\alpha_1 \ln(10) d + O(\ln(d)) \simeq 2.07d. \tag{12}$$

In analyzing the time required by Algorithm B1 and other algorithms described in Section 4, we make the following simplifying assumptions.

(a) Only the time required for the inner loop(s) is considered. (The computation of $\ln(n)$ is common to all the algorithms considered, so the time required for this is neglected. The final division of U_K by V_K takes time $O(d^2)$ if done as in [9], but the constant factor is relatively small, and $o(d^2)$ methods exist [5].)

(b) Multiplication or division of a multiple-precision number (e.g. A_{k-1}) by a small integer (e.g. n^2 or k) takes time d units. In the analysis (though not in the implementation of the algorithm) the possibility of reducing the working precision (e.g. for A_k and B_k when $k \simeq K$) is neglected. Considering this possibility complicates the analysis but is unlikely to alter the ranking of the algorithms discussed below.

(c) Addition of two multiple-precision numbers takes time d units. (The constant is assumed to be the same as for (b). Again, this is unlikely to change the ranking of the algorithms.)

Using these assumptions, each iteration of (10) requires time $8d$ (for 3 multiple-precision additions, 2 multiplications and 3 divisions by small integers). Thus, from (12), the time required by method B1 is about $2\alpha_1 \ln(10)d^2 \simeq 16.5d^2$.

It is important to note that we avoid keeping H_k or $(H_k - \ln(n))$ as a multiple-precision number and multiplying by B_k in the inner loop. This would lead to a method with time $\Omega(d^3)$ if the classical multiplication algorithm were used as in [9]. The idea of using the Bessel function identities to compute γ was suggested by the second author, and the $O(d^2)$ implementation was discovered by the first author.

If terms in the sum (15) are grouped as in [5] and the Schönhage-Strassen fast multiplication algorithm [15] is used, it is possible to compute γ with error bounded by 10^{-d} in time $O\{d[\ln(d)]^3 \ln[\ln(d)]\}$, asymptotically faster than any of the $\Omega(d^2)$ algorithms considered here. However, such "fast" algorithms are very difficult to implement and are slower than Algorithm B1 unless d is very large. Thus, we do not consider them further.

4. Related Algorithms. In this section we briefly describe and compare several closely related algorithms for the computation of γ.

4.1. *Algorithm* B2. From (2) we have

$$\gamma + \ln(n) = \frac{S_0(2n) - K_0(2n)}{I_0(2n)},$$

and from (3)

$$K_0(2n) = \tfrac{1}{2}(\pi/n)^{1/2} e^{-2n} \sum_{k=0}^{4n} (-1)^k a_k(2n) + O(e^{-6n}/n). \tag{13}$$

Thus, we can find γ with error $O(n^{-1/2}e^{-8n})$ if $K_0(2n)$ is approximated using (13). If e^{2n} is computed using the Taylor series, and the time required to compute $(\pi/n)^{1/2}$ is neglected, the time required by this method (B2) is about

$$[(8\alpha_3 + 3\alpha_1 + 6)\ln(10)/8]d^2 \simeq 16.3d^2,$$

not appreciably less than for the simpler method B1.

4.2. *Algorithm* B3. To avoid the computation of $(\pi/n)^{1/2}e^{-2n}$ in (13), we may use the asymptotic series [1, Eq. (9.7.5)]

$$I_0(2n)K_0(2n) \sim \frac{1}{4n} \sum_{k=0}^{k'} \frac{[(2k)!]^3}{(k!)^4 (16n)^{2k}} \tag{14}$$

with $k' \leqslant 2n$. Empirical evidence suggests that the relative error in (14) with $k' = 2n$ is $O(n^{-1/2}e^{-4n})$, but we have not been able to prove this. Assuming this error bound, the time required with $k' = 2n$ is about

$$(\alpha_3 + 3/8)\ln(10)d^2 \simeq 12.3d^2.$$

This is less than the estimate $16.5d^2$ for Algorithm B1, but we preferred to use B1 because of its simplicity and the difficulty in rigorously bounding the error in (14).

4.3. *Exponential Integral Methods.* Several algorithms are based on the identity

$$\gamma + \ln(n) = Q(n) - R(n),$$

where

$$Q(n) = \int_0^n \left(\frac{1 - e^{-x}}{x}\right) dx = \sum_{k=1}^{\infty} \frac{n^k(-1)^{k-1}}{k!k} \tag{15}$$

and

$$R(n) = \int_n^{\infty} \frac{e^{-x}}{x} dx = O(e^{-n}/n).$$

Beyer and Waterman [3], [4] took $n \simeq \ln(10)d$, worked to precision equivalent to $2d$ decimal places to compensate for cancellation in the sum for $Q(n)$, and neglected $R(n)$. The time required for this method is about $6\alpha_1 \ln(10)d^2 \simeq 49.6d^2$, or three times that for method B1.

Following the suggestion of Sweeney [16], Brent [7], [8] took $n \simeq \frac{1}{2}\ln(10)d$, summed the series for $Q(n)$ using the equivalent of $3d/2$ decimal places, and approximated $R(n)$ by its asymptotic expansion

$$R(n) = \frac{e^{-n}}{n} \sum_{k=0}^{n-2} \frac{k!}{(-n)^k} + O(e^{-2n}/n). \tag{16}$$

Assuming the power series is used to compute e^n, the time required for this method is about

$$\frac{3}{4}(3\alpha_2 + \alpha_0 + 1)\ln(10)d^2 \simeq 28.8d^2,$$

about twice as much as for method B1. Actual running times confirm this ratio.

Using the identity

$$e^n Q(n) = \sum_{k=0}^{\infty} H_k \frac{n^k}{k!},$$

we can evaluate $Q(n)$ by computing $\Sigma H_k n^k/k!$ and $\Sigma n^k/k!$ by recurrences similar to (10). Because all the terms in the two sums are positive, there is no need to increase the working precision to much more than the equivalent of d decimal places. If $n \simeq \ln(10)d$, and $R(n)$ is neglected, the time required is about $7\alpha_0 \ln(10)d^2 \simeq 43.8d^2$, slightly less than for Beyer and Waterman's method. If the asymptotic series (16) is used for $R(n)$, the time required is about

$$\tfrac{1}{4}(14\alpha_1 + 3)\ln(10)d^2 \simeq 30.7d^2,$$

slightly greater than for Sweeney's method.

Instead of using the asymptotic expansion (16) for $R(n)$, we could use Euler's continued fraction [18, p. 350]

$$e^n R(n) = 1/n + 1/1 + 1/n + 2/1 + 2/n + 3/1 + 3/n + \cdots$$

and the forward or backward recurrence relations. This has the advantage that $R(n)$ can be evaluated as accurately as desired, whereas with the asymptotic expansion (16) the error is $\Omega(e^{-2n}/n)$. The choice of the optimal n and the optimal number of terms in the continued fraction (evaluated by the backward recurrence relations) gives a method competitive with Algorithm B1, but much more complicated.

4.4. *A Generalization.* For fixed $p > 0$, it follows from (1) that

$$\gamma = \lim_{n\to\infty} \frac{\sum_{k=0}^{\infty} (n^k/k!)^p (H_k - \ln(n))}{\sum_{k=0}^{\infty} (n^k/k!)^p}. \tag{17}$$

With $p = 1$ we obtain essentially one of the exponential integral methods mentioned above, with error $O(e^{-n}/n)$. With $p = 2$ we obtain method B1, with error $O(e^{-4n})$.

We shall sketch how the error in (17) may be estimated for integer $p \geqslant 2$. Let $y(z)$ be a function of the real variable z, and L the operator defined by $Ly = z(dy/dz)$. Then

$$V_p(z) = \sum_{k=0}^{\infty} (z/k!)^p$$

and

$$U_p(z) = \sum_{k=0}^{\infty} (H_k - \ln(z) - \gamma)(z^k/k!)^p$$

are independent solutions of

$$(L^p - (pz)^p)y = 0. \tag{18}$$

{To verify this for $y = U_p$, let

$$V_{p,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^{kp+\nu}}{(k!)^{p-1}\Gamma(k+\nu+1)}.$$

Then it is easy to check that

$$[(L + (p-1)\nu)(L-\nu)^{p-1} - (pz)^p]\, V_{p,\nu}(z) = 0, \tag{19}$$

and the result follows by differentiating (19) with respect to ν, setting $\nu = 0$, and observing that $U_p(z) = -\partial V_{p,\nu}(z)/\partial\nu|_{\nu=0}$.} Now

$$V_p(z) \sim p^{-1/2}(2\pi z)^{(1-p)/2}\exp(pz)$$

is the dominant solution of (18) as $z \longrightarrow +\infty$. By analyzing the asymptotic behavior of the subdominant solutions we obtain

$$|U_p(z)/V_p(z)| = O(\exp(-c(p)z)) \quad \text{as } z \longrightarrow +\infty,$$

where $c(p) = p(1 - \cos(2\pi/p))$. Thus, the error in (17) for integer $p \geqslant 2$ is $O(\exp(-c(p)n))$ as $n \longrightarrow \infty$. Since $c(2) = 4$, $c(3) = 4.5$, and $c(p) \leqslant 4$ for $p \geqslant 4$, only the case $p = 3$ is worth considering as a computational alternative to method B1 (i.e. the case $p = 2$).

5. Computational Results. γ was computed to more than 30,100 decimal places using method B1 and a multiple-precision arithmetic package [9] on a Univac 1100/42. Three independent computations were performed, with $n = 17{,}332$ (using base 10,000 and 7,527 digits), $n = 17{,}357$ (base 65,535 and 6,260 digits), and $n = 17{,}387$ (base 65,536 and 6,271 digits). All three agreed to 30,100 decimal places, and the last two agreed to 30,141 decimal places. The computer time required for each computation was about 20 hours, much the same as for the 20,700 decimal place computation [7] using Sweeney's method on the same machine.

We also computed $G = \exp(\gamma)$ to more than 30,100 decimal places using the exponential routine in Brent's package [9] (with base 65,536 and 6,260 digits), and verified it by computing $\ln(G)$ by the Gauss-Salamin algorithm [6] (with base 10,000 and 7,550 digits). The rounded 30,100D values of γ and G are given in [10].

The first 29,200 partial quotients in the regular continued fractions for γ and G were computed and verified as in [7], [8]. Statistics on the distributions of the first 29,000 partial quotients are given in Table 1, with notation as in [7, Table 2]. A chi-squared test did not show any significant difference (at the 5% level) between the actual distributions and the distribution predicted by the Gauss-Kusmin theorem [13]. A table of the first 29,000 partial quotients for γ and G is given in [11].

TABLE 1

Distribution of first 29,000 partial quotients for γ and G

n	number of $q_i(\gamma) = n$	number of $q_i(G) = n$	expected number
1	12112	11992	12036.1
2	4809	4875	4927.8
3	2791	2760	2700.2
4	1727	1757	1707.9
5	1181	1168	1178.6
6	867	848	862.7
7	642	716	658.9
8	497	520	519.7
9	420	417	420.5
10	346	335	347.2
11–20	1624	1729	1694.1
21–50	1148	1103	1133.9
51–100	411	390	400.2
101–1000	378	349	370.4
>1000	47	41	41.8

From the continued fractions for γ and G we can improve the Theorem of [7, Section 7], where the lower bound on $|Q|$ was $10^{10,000}$.

THEOREM 1. *If γ or $G = P/Q$ for integers P and Q, then $|Q| > 10^{15,000}$.*

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